Lebesgue measurability and large cardinals

1 Introduction

Lebesgue measure is motivated by the following idea. It is clear that by the "length" of an interval (a, b) or $[a, b]$ we mean $b - a$. Can we extend this notion to many other subsets of $\mathbb{R}$ in a natural way? In other words, can we find some $\mu : S \to [0, \infty]$ for some "large" subset $S \subseteq \mathcal{P}(\mathbb{R})$ with the following properties?

1. if $a, b \in \mathbb{R}$ with $a < b$, then $\mu((a, b)) = \mu([a, b]) = b - a$
2. (*monotonicity*) if $A, B \in S$ with $A \subseteq B$, then $\mu(A) \leq \mu(B)$
3. (*countable additivity*) if $(A_i)_{i \in I}$ is a countable collection of pairwise disjoint members of S, then $\mu\left(\bigcup_{i \in I} A_i\right) = \sum_{i \in I} \mu(A_i)$
4. (*translation invariance*) if $x \in \mathbb{R}$ and $A \in S$, then $\mu(\{x + a : a \in A\}) = \mu(A)$

Definition. The *outer Lebesgue measure* of a set $A \subseteq \mathbb{R}$ is

$$\mu^*(A) := \inf\left\{\sum_{i=0}^{\infty}(b_i - a_i) : a_i, b_i \in \mathbb{R} \text{ and } a_i < b_i \ \forall i \in \omega, \text{ and } A \subseteq \bigcup_{i=0}^{\infty}(a_i, b_i)\right\}.$$

A is (*Lebesgue*) *measurable* iff $\mu^*(A) = \mu^*(A \cap B) + \mu^*(A \cap (\mathbb{R} \setminus B))$ for every $B \subseteq \mathbb{R}$, in which case the *Lebesgue measure* of A is $\mu(A) := \mu^*(A)$.

It can be shown that Lebesgue measure satisfies properties 1–4, and that many naturally-occurring sets are Lebesgue measurable, such as Borel sets (see section 2.3). For the proof we refer the reader to Carathéodory's extension theorem.

However, the following construction shows that we can never achieve $S = \mathcal{P}(\mathbb{R})$. In particular, not every set of reals is Lebesgue measurable.

Example (Vitali). Define the equivalence relation $\sim$ on $\mathbb{R}$ by $x \sim y$ iff $y - x \in \mathbb{Q}$. By the axiom of choice, let $A \subseteq [0, 1]$ contain exactly one member from each equivalence class, and suppose for contradiction $A \in S$.

Write $A_q = \{a + q : q \in Q\}$ for each $q \in Q$, where $Q = \mathbb{Q} \cap [-1, 1]$. Then by translation invariance, $\mu(A_q) = \mu(A)$ for each $q \in Q$. Now $(A_q)_{q \in Q}$ is a collection of pairwise disjoint sets, so by countable additivity $\mu\left(\bigcup_{q \in Q} A_q\right) = \sum_{q \in Q} \mu(A_q) = \sum_{q \in Q} \mu(A) = 0$ or ∞.

But $[0, 1] \subseteq \bigcup_{q \in Q} A_q \subseteq [-1, 2]$, so by monotonicity $1 \leq \mu\left(\bigcup_{q \in Q} A_q\right) \leq 3$. Contradiction.

This construction uses the axiom of choice (AC). Thus we can still hope that, in some sense, every explicitly describable set of reals is measurable.

It may be natural to ask: is it consistent with ZF that every set of reals is Lebesgue measurable? But this question is not really appropriate, since AC is used in the proof that Lebesgue measure is countably additive. Indeed, if ZF is consistent then it is consistent with ZF that $\mathbb{R}$ is a countable union of countable sets [10, p. 142], in which case if $S = \mathcal{P}(\mathbb{R})$ then countable additivity implies $\mu(\mathbb{R}) = 0$!

However, only a weak form of AC is required to prove the basic properties of Lebesgue measure. The following statement follows from AC via Zorn's lemma, and implies the axiom of countable choice.

Definition. The *axiom of dependent choice* (DC) is the statement that if X is a non-empty set and R is a binary relation on X such that for all $x \in X$ there exists $y \in X$ with xRy, then we can choose a sequence $(x_n)_{n\in\omega}$ from X with $x_0 R x_1 R x_2 R \dots$.

This axiom is sufficient to develop the theory of Lebesgue measure and for most of real analysis. By contrast, it is not enough for Vitali's example: since each equivalence class was countable, we made $2^{\aleph_0}$ choices, each between $\aleph_0$ options.

The purpose of this essay, then, is to address the following.

Question. Is it consistent with $\mathsf{ZF} + \mathsf{DC}$ that every set of reals is Lebesgue measurable?

2 Solovay's model

Surprisingly, this question is connected to the existence of a strongly inaccessible cardinal.

Notation. Denote by LM the statement that every set of reals is Lebesgue measurable.

Denote by I the statement that there is a strongly inaccessible cardinal.

This chapter is devoted to proving the following positive answer to the above question.

Theorem 1. *If* $\mathsf{ZFC} + \mathsf{I}$ *is consistent, then* $\mathsf{ZF} + \mathsf{DC} + \mathsf{LM}$ *is consistent.*

To prove this, we will use forcing to prove the following in ZFC.

Theorem 2 (Solovay [4]). *Suppose there is a countable transitive $\in$-model of* $\mathsf{ZFC} + \mathsf{I}$*. Then there is a countable transitive $\in$-model of* $\mathsf{ZF} + \mathsf{DC} + \mathsf{LM}$*.*

To prove Theorem 1 from this, we follow the approach of Kunen [1, p. 245]. This requires a modified version of Theorem 2, which will easily be seen to follow from the same proof. The following is in fact a theorem scheme rather than a theorem of ZFC (like the reflection principle).

Theorem 2′. *For each finite $\Omega \subseteq \mathsf{ZF}$, there is a finite $\Lambda \subseteq \mathsf{ZFC}$ such that the following is a theorem of* ZFC*.*

Suppose there is a countable transitive $\in$-model of $\Lambda + \mathsf{I}$. Then there is a countable transitive $\in$-model of $\Omega + \mathsf{DC} + \mathsf{LM}$.

Proof of Theorem 1 from Theorem 2′. Suppose $\mathsf{ZF} + \mathsf{DC} + \mathsf{LM}$ is inconsistent. Then $\Omega + \mathsf{DC} + \mathsf{LM}$ is inconsistent for some finite $\Omega \subseteq \mathsf{ZF}$, and moreover this fact is provable in ZFC since it is witnessed by a finite proof of $\perp$. Choose a corresponding finite $\Lambda \subseteq \mathsf{ZFC}$ as in Theorem 2′.

In this paragraph we work in $\mathsf{ZFC} + \mathsf{I}$. By the reflection principle, $\Lambda + \mathsf{I}$ has a $\in$-model, which by the downward Löwenheim–Skolem theorem has a countable elementary submodel. This is a countable $\in$-model of $\Lambda + \mathsf{I}$, but it need not be transitive. However, since it satisfies the axiom of extensionality and $\in$ is well-founded, we may apply the Mostowski collapse lemma to obtain an isomorphic transitive $\in$-model. This is a countable transitive $\in$-model of $\Lambda + \mathsf{I}$, so by Theorem 2′ there is a countable transitive $\in$-model of $\Omega + \mathsf{DC} + \mathsf{LM}$. So by the soundness theorem, $\Omega + \mathsf{DC} + \mathsf{LM}$ is consistent.

Thus it is possible in $\mathsf{ZFC} + \mathsf{I}$ to prove both the consistency and the inconsistency of the theory $\Omega + \mathsf{DC} + \mathsf{LM}$. So $\mathsf{ZFC} + \mathsf{I}$ is inconsistent. □

Outline of proof of Solovay's theorem. Working in ZFC, we will begin with a countable transitive $\in$-model M of $\mathsf{ZFC} + \mathsf{I}$ in which (κ is a strongly inaccessible cardinal)^M.

We will then use a forcing poset known as the "Lévy collapse" to pass to a forcing extension $M[G]$ in which all ordinals smaller than κ are "collapsed" to ω.

Finally, we will consider the submodel $HOD^*_{M[G]}$ given by those sets that are "hereditarily ordinal-sequence-definable in $M[G]$", and show that this is a transitive $\in$-model of $\mathsf{ZF} + \mathsf{DC} + \mathsf{LM}$.

For the rest of this chapter, then, we work in ZFC unless stated otherwise.

2.1 Ordinal definability

Although in the proof of Solovay's theorem we will need the notion of definability by an infinite sequence of ordinals, it will be instructive to first consider the notion of definability by finitely many ordinals. We use ideas from Kunen [1, pp. 145–147].

In this section, assume that M is a transitive $\in$-model of ZF. By this we mean in particular that M is a set.

Definition. We say $a \in M$ is *ordinal definable in M*, and write $a \in OD_M$, iff there exists a formula $\phi(x_1, x_2, \ldots, x_n, y)$[1] in the language of set theory and $\alpha_1, \alpha_2, \ldots, \alpha_n \in On^M$ such that $\phi(\alpha_1, \alpha_2, \ldots, \alpha_n, a)^M$ and a is unique in M with this property.

We say $a \in M$ is *hereditarily ordinal definable in M*, and write $a \in HOD_M$, iff $\mathrm{TC}(\{a\}) \subseteq OD_M$.

Proposition 1. *HOD_M is a transitive $\in$-model of* ZF.

The proof of Proposition 1 will be very similar to that of the corresponding result for forcing extensions, in the following sense. Since HOD_M is transitive, in several cases, the fact that an axiom holds in HOD_M will follow from the fact that it holds in M, together with a simple formula provided by the axiom itself to demonstrate ordinal definability in M. The difficulty will arise when we reach the axiom scheme of separation, and concerns the definability of OD_M in M.

To illustrate the difficulty, let $b \in HOD_M$ and $\phi(x)$ be a formula in the language of set theory, and let $c = \left\{d \in b : \phi(d)^{HOD_M}\right\}$. Why should $c \in OD_M$? We would like to define c from b by the formula $\forall x \left(x \in c \Leftrightarrow \left(\psi(x) \wedge \phi(x)^{HOD_M}\right)\right)$, where $(\forall x (\psi(x) \Leftrightarrow x \in b))^M$. But suppose $\phi(x)$ has the form $\forall y \varphi(x, y)$. Then $\phi(x)^{HOD_M}$ has the form $(\forall y \in HOD_M)\left(\varphi(x,y)^{HOD_M}\right)$, i.e. $\forall y \left(\mathrm{TC}(\{y\}) \subseteq OD_M \Rightarrow \varphi(x,y)^{HOD_M}\right)$. How do we know that there is a formula $\chi(x)$ such that for every $a \in M$, $a \in OD_M$ iff $\chi(a)^M$? We certainly can't express the above definition of OD_M in this fashion directly without first constructing a formula $\xi(x)$ such that, for every formula ζ in the language of set theory, $(\xi(\ulcorner\zeta\urcorner) \Leftrightarrow \zeta)^M$, which would contradict Tarski's undefinability theorem. We nonetheless have the following result, which is analogous to the definability lemma for forcing extensions.

[1] It is implicit that the free variables of a formula $\phi(x_1, x_2, \ldots, x_n)$ are exactly $x_1, x_2, \ldots, x_n$.

Lemma 1. *There exists a formula* $\chi(x)$ *in the language of set theory such that for every* $a \in M$*,*

$$a \in OD_M \text{ iff } \chi(a)^M .$$

The key idea of Lemma 1 is to use the reflection principle in M.

Working in ZF, if $\beta \in On$ then V_β is a set. Therefore using model theory we may express by a formula $\mathrm{DEF}\left(\beta, \ulcorner\phi(x_1, x_2, \ldots, x_n, y)\urcorner, (\alpha_1, \alpha_2, \ldots, \alpha_n), a\right)$ the property that: $\beta \in On$; $a \in V_\beta$; $\phi(x_1, x_2, \ldots, x_n, y)$ is a formula in the language of set theory; $\alpha_1, \alpha_2, \ldots, \alpha_n \in On \cap V_\beta$; $\phi(\alpha_1, \alpha_2, \ldots, \alpha_n, a)^{V_\beta}$; and a is unique in V_β with this property.

Let $\chi(x)$ be the formula

$$\exists\beta\exists m\exists\alpha \mathrm{DEF}(\beta, m, \alpha, x) .$$

Proof of "if". Suppose $a \in M$ with $\chi(a)^M$. Then there exist $\beta, m, \alpha \in M$ with $\mathrm{DEF}(\beta, m, \alpha, a)^M$ and, by definition of DEF, a is unique in M with this property. Write $m = \ulcorner\phi(x_1, x_2, \ldots, x_n, y)\urcorner$ and $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$. Then we can define a in M using DEF together with $\beta, m, \alpha_1, \alpha_2, \ldots, \alpha_n$, so it is sufficient to prove that $m \in On^M$. But without loss of generality our system for encoding formulas is such that $m \in \omega^M$. □

Proof of "only if". Suppose $a \in OD_M$, say by the formula $\phi(x_1, x_2, \ldots, x_n, y)$ and $\alpha_1, \alpha_2, \ldots, \alpha_n \in On^M$. Since M is a model of ZF, we may apply the reflection principle to obtain $\beta \in On^M$ with $\alpha_1, \alpha_2, \ldots, \alpha_n < \beta$ and $a \in (V_\beta)^M$ such that $\left(\phi(\alpha_1, \alpha_2, \ldots, \alpha_n, a)^{V_\beta} \text{ and } a \text{ is unique in } V_\beta \text{ with this property}\right)^M$. Since $\alpha_1, \alpha_2, \ldots, \alpha_n \in (V_\beta)^M$, it follows that $\mathrm{DEF}\left(\beta, \ulcorner\phi(x_1, x_2, \ldots, x_n, y)\urcorner, (\alpha_1, \alpha_2, \ldots, \alpha_n), a\right)^M$ and hence $\chi(a)^M$. □

In view of Lemma 1, we define the classes $OD = \{x : \chi(x)\}$ and $HOD = \{x : \mathrm{TC}(\{x\}) \subseteq OD\}$, so that by Lemma 1, $OD_M = OD^M$ and so $HOD_M = HOD^M$, since the formula $y \in \mathrm{TC}(\{x\})$ is absolute for M.

We proceed to the proof of Proposition 1. Most of this should be viewed by the reader as a sequence of trivial checks: after all, we are checking that set-theoretic operations defined by certain formulas can be carried out in HOD_M, which is the kind of thing specifically allowed by definition OD_M; and since $HOD_M = HOD^M$, any difficulties of the character described above will be solved by relativising a formula to HOD.

Proof of Proposition 1. First note that since M is transitive and HOD_M is defined as a subset of M by a hereditary property, HOD_M is transitive, and it is a $\in$-model by definition. So the axioms of extensionality and foundation hold in M.

Next, clearly $On^M \subseteq HOD_M$, so by the axioms of empty set and infinity in M and absoluteness, these axioms also hold in M.

For the axiom of pairing, suppose $a, b \in HOD_M$. By the axiom of pairing in M, $\{a, b\} \in M$. Moreover, the axiom itself provides us with a formula defining $\{a, b\}$ in M from a and b, so $\{a, b\} \in OD_M$ and hence $\{a, b\} \in HOD_M$. The axiom of union follows similarly.

It is only the remaining axioms (power set, separation and replacement) that require Lemma 1. For the axiom of power set, let $a \in HOD_M$, say defined in M by the formula $\phi(x_1, x_2, \ldots, x_n, y)$ and $\alpha_1, \alpha_2, \ldots, \alpha_n \in On^M$. It is sufficient to prove that $\mathcal{P}(a) \cap HOD_M \in$

OD_M. First of all, by Lemma 1 this set is in M by applying the axiom scheme of separation in M to $\mathcal{P}(a)^M = \mathcal{P}(a) \cap M$. Then, by Lemma 1, $\mathcal{P}(a) \cap HOD_M$ is defined in M by the formula $\forall z \, (z \in y \Leftrightarrow \exists w \, (\phi(x_1, x_2, \ldots, x_n, w) \wedge z \subseteq w \wedge z \in HOD))$ and $\alpha_1, \alpha_2, \ldots, \alpha_n$.

For the axiom scheme of separation, we follow the discussion preceding Lemma 1. Let $b \in HOD_M$ and $\phi(x)$ be a formula in the language of set theory, and let $c = \left\{d \in b : \phi(d)^{HOD_M}\right\}$. By Lemma 1, the formula $\phi(x)^{HOD_M}$ is equivalent to $\left(\phi(x)^{HOD}\right)^M$. So by the axiom scheme of separation in M, $c \in M$, and the axiom itself together with $\phi(x)^{HOD}$ provide us with a formula defining c in M. So $c \in OD_M$ and hence $c \in HOD_M$. The axiom scheme of replacement follows similarly by relativising a formula defining a function in HOD_M to HOD. □

Remarks.

1. In accordance with the modified version of Solovay's theorem (Theorem 2′), in verifying any given axiom of ZF in HOD_M we only used finitely many axioms of ZF in M. (Although Lemma 1 used infinitely many instances of the reflection principle for M, only finitely many of them were ever needed for any single axiom.)

2. HOD_M is also a model of AC, even if M is not [1, p. 147].

3. We have been working with a set model of ZF. However, even though HOD is a proper class, we could easily adapt the proof that HOD_M is a model of ZFC to prove the following theorem scheme: for each axiom ϕ of ZFC, ϕ^{HOD} is a theorem of ZF. This provides us with a proof of the consistency of ZFC relative to ZF, since it shows that if ψ is a theorem of ZFC, then ψ^{HOD} is a theorem of ZF by relativising the proof to HOD.

At last we introduce the notion needed in the proof of Solovay's theorem.

Definition. We say $a \in M$ is *ordinal-sequence-definable in M*, and write $a \in OD^*_M$, iff there exists a formula $\phi(x, y)$ in the language of set theory and $f \in M$ with $f : \omega \to On^M$ such that $\phi(f, a)^M$ and a is unique in M with this property.

We say $a \in M$ is *hereditarily ordinal-sequence-definable in M*, and write $a \in HOD^*_M$, iff $\mathrm{TC}(\{a\}) \subseteq OD^*_M$.

All we will need later from this section are the following two results.

Theorem 3. *Suppose in addition that M is a model of* DC. *Then HOD^*_M is a transitive $\in$-model of* ZF + DC.

To prove this, we follow the approach of Jech [2, pp. 519–520] and use the lemma that follows. Later, this lemma will allow us to show that every real and every Borel set of M lies in HOD^*_M.

Lemma 2. *Suppose in addition that M is a model of* DC, *and let $g \in M$ be a function on ω with values in OD^*_M. Then $g \in OD^*_M$.*

Proof of Theorem 3 from Lemma 2. The proof that HOD_M^* is a transitive $\in$-model of ZF is almost identical to that for HOD_M.

For DC, let $a, R \in HOD_M^*$ satisfy the hypotheses of DC. By DC in M, there exists a function $g \in M$ on ω with values in a such that $g(0)Rg(1)Rg(2)R\dots$. By Lemma 2, $g \in OD_M^*$ and hence $g \in HOD_M^*$. □

Proof of Lemma 2. If we were to replace ω in Lemma 2 by $n \in \omega$, then it would become trivial by taking a finite conjunction of formulas. In place of an "infinite conjunction", we must quantify over ω. Thus we would like to somehow have a single formula with which we can define in M every member of OD_M^* by a different sequence of ordinals in M. To achieve this we delve further into the ideas of Lemma 1.

Recall the definition of DEF used to define $\chi(x)$ for Lemma 1, and define DEF^* analogously. Then by the proof of "only if" in Lemma 1, for every $a \in OD_M^*$ there exist $\beta \in On^M$, $m \in On^M$[2] and $f : \omega \to On^M$ such that $\mathrm{DEF}^*(\beta, m, f, a)^M$ and, by definition of DEF^*, a is unique in M with this property. Denote by f' the sequence obtained by appending β and m to the front of f, and define the formula $\psi(x, y)$ so that $\psi(f', y)$ is equivalent to $\mathrm{DEF}^*(\beta, m, f, y)$.

Thus we have a single formula $\psi(x, y)$ with which we can define in M every $a \in OD_M^*$ by some sequence f'_a of ordinals in M. Turning now to g, by the axiom of countable choice (which follows from DC) in M, the "2-dimensional sequence" of ordinals $\left(f'_{g(n)}(k)\right)_{(n,k)\in\omega\times\omega}$ lies in M. We can now use ψ and quantification over ω to define g in M by this "2-dimensional sequence". To show that g is definable in M by a "1-dimensional sequence" of ordinals in M, we need only observe that M contains a bijection $\omega \times \omega \to \omega$, a notion which is absolute for M. □

2.2 The Lévy collapse

The Lévy collapse is the type of forcing poset used in the proof of Solovay's theorem. We assume the reader is familiar with the basics of forcing from Part III Set Theory, but adopt the convention that forcing posets are ordered so as to have maximal (rather than minimal) elements. We use ideas from Kanamori [3, pp. 127–129, 140].

In this section, assume that M is a countable transitive $\in$-model of ZFC.

Recall the forcing poset $\mathrm{Fn}(\lambda, \kappa) := \{p \in [\lambda \rightharpoonup \kappa] : |p| < \aleph_0\}$,[3] ordered by $p \leq q$ iff $p \supseteq q$, where $\lambda, \kappa \in M$ with (λ and κ are infinite cardinals)^M and $\lambda < \kappa$ as ordinals. Recall [1, p. 263] that if G is a $\mathrm{Fn}(\lambda, \kappa)$-generic filter over M,[4] then $\bigcup G$ is a surjection $\lambda \to \kappa$, so that κ is "collapsed" onto λ in $M[G]$. The Lévy collapse is a generalisation of this idea allowing us to collapse multiple ordinals at once.

Definition. Let λ be a regular cardinal and $S \subseteq On$. The *Lévy collapse*[5] $\mathrm{Col}(\lambda, S)$ is the

[2] Again we assume without loss of generality that formulas are encoded as members of ω.

[3] $[\lambda \rightharpoonup \kappa]$ denotes the set of partial functions from λ to κ, viewed as a set of ordered pairs.

[4] Of course by "$\mathrm{Fn}(\lambda, \kappa)$-generic over M" we mean the same as "$\mathrm{Fn}(\lambda, \kappa)^M$-generic over M".

[5] Other authors reserve the term *Lévy collapse* for the special case in which (S is a cardinal)^M and $\lambda < S$ as ordinals.

forcing poset

$$\left\{p \in \prod_{\alpha \in S} [\lambda \rightharpoonup \alpha] : |p| < \lambda\right\}.$$

View this product as a subset of $[S \times \lambda \rightharpoonup \bigcup S]$, and order $\mathrm{Col}(\lambda, S)$ by $p \leq q$ iff $p \supseteq q$.

In what follows, let $\lambda \in M$ with $(\lambda \text{ is a regular cardinal})^M$ and $S \in M$ with $S \subseteq On^M$.

We begin by checking that the ordinals in S are indeed collapsed to λ when forcing with $\mathrm{Col}(\lambda, S)^M$.

Proposition 2. *Let G be a $\mathrm{Col}(\lambda, S)$-generic filter over M. Then for every $\alpha \in S$, $(|\alpha| \leq \lambda)^{M[G]}$.*

Proof. Fix $\alpha \in S \setminus \{0\}$. Let G_α be the projection of G onto the factor $[\lambda \rightharpoonup \alpha]$ in the above product, and let $f_\alpha = \bigcup G_\alpha$. Clearly $f_\alpha \in M[G]$, and $f_\alpha \in [\lambda \rightharpoonup \alpha]$ since G is directed. We show that f_α is a surjection $\lambda \to \alpha$.

To see that f_α is total, let $x \in \lambda$. Then $\left\{p \in \mathrm{Col}(\lambda, S)^M : (\alpha, x) \in \mathrm{dom}(p)\right\}$ is a dense subset of $\mathrm{Col}(\lambda, S)^M$ lying in M, so $x \in \mathrm{dom}(f_\alpha)$.[6]

To see that f_α is surjective, let $y \in \alpha$. Then $\left\{p \in \mathrm{Col}(\lambda, S)^M : (\exists x \in \lambda)(y = p(\alpha, x))\right\}$ is a dense subset of $\mathrm{Col}(\lambda, S)^M$ lying in M, so $(\exists x \in \lambda)(y = f_\alpha(x))$. □

We now introduce a fundamental property of the Lévy collapse. The following definition and proposition are based on Kanamori [3, p. 129] and Kunen [1, pp. 255, 275].

Definition. A forcing poset P is called *weakly homogeneous* iff for every $p, q \in P$ there is an order-automorphism π of P such that $\pi(p)$ and q are compatible.

Proposition 3.

1. *(*zero-one law*) Suppose $P \in M$ is a forcing poset with maximal element 1 and (P is weakly homogeneous)^M. Let $\phi(x_1, x_2, \ldots, x_n)$ be a formula in the language of set theory and $a_1, a_2, \ldots, a_n \in M$. Then either $1 \Vdash \phi(\check{a}_1, \check{a}_2, \ldots, \check{a}_n)$[7] or $1 \Vdash \neg\phi(\check{a}_1, \check{a}_2, \ldots, \check{a}_n)$.*

2. *($\mathrm{Col}(\lambda, S)$ is weakly homogeneous)^M.*

Proof.

1. First we claim that if $p \in P$ with $p \Vdash \phi(\check{a}_1, \check{a}_2, \ldots, \check{a}_n)$ and π is an order-automorphism of P, then $\pi(p) \Vdash \phi(\check{a}_1, \check{a}_2, \ldots, \check{a}_n)$. To see this, let H be a P-generic filter over M containing $\pi(p)$. Then $\pi^{-1}(H)$ is a P-generic filter over M containing p, so $\phi(a_1, a_2, \ldots, a_n)^{M[\pi^{-1}(H)]}$. Now π induces a bijection $M^P \to M^P$ fixing $\check{a}$ for every $a \in M$, which in turn induces a bijection $M[\pi^{-1}(H)] \to M[H]$ fixing M pointwise, and since π is an order-automorphism this bijection is a $\in$-isomorphism. It follows that $\phi(a_1, a_2, \ldots, a_n)^{M[H]}$. This proves the claim.

[6] Note that a P-generic filter intersects every dense subset of P (not just the open dense subsets) [1, p. 269].

[7] $\check{a}$ denotes the canonical P-name for $a \in M$.

Now suppose for contradiction neither $1 \Vdash \phi(\check{a}_1, \check{a}_2, \ldots, \check{a}_n)$ nor $1 \Vdash \neg\phi(\check{a}_1, \check{a}_2, \ldots, \check{a}_n)$. Then there must be P-generic filters H_1, H_2 over M with $\phi(a_1, a_2, \ldots, a_n)^{M[H_1]}$ and $\neg\phi(a_1, a_2, \ldots, a_n)^{M[H_2]}$. So by the truth lemma for forcing extensions there exist $p, q \in P$ with $p \Vdash \phi(\check{a}_1, \check{a}_2, \ldots, \check{a}_n)$ and $q \Vdash \neg\phi(\check{a}_1, \check{a}_2, \ldots, \check{a}_n)$.

Now by weak homogeneity, let π be an order-automorphism of P such that $\pi(p)$ and q are compatible, say with common lower bound r. Let H be a P-generic filter over M containing r, so that $\pi(p), q \in H$. Then by the claim, $\pi(p) \Vdash \phi(\check{a}_1, \check{a}_2, \ldots, \check{a}_n)$. But $q \Vdash \neg\phi(\check{a}_1, \check{a}_2, \ldots, \check{a}_n)$, so no P-generic filter over M can contain both $\pi(p)$ and q. This contradicts $\pi(p), q \in H$.

2. We carry out the proof inside M. Let $p, q \in \mathrm{Col}(\lambda, S)$. Define $X_p = \{x \in \lambda : (\exists \alpha \in S)((\alpha, x) \in \mathrm{dom}(p))\}$ and define X_q similarly. Then $|X_p|, |X_q| < \lambda$ since $|p|, |q| < \lambda$, so there exists a bijection $f : \lambda \to \lambda$ such that $f(X_p)$ is disjoint from X_q. This induces an order-automorphism π of P such that $\mathrm{dom}(\pi(p))$ is disjoint from $\mathrm{dom}(q)$, which implies that $\pi(p)$ and q are compatible. So π is as required. □

Having established these general properties of the Lévy collapse, we now turn to the special case to be used in the proof of Solovay's theorem, in which $\lambda = \omega$ and S is a strongly inaccessible cardinal in M. In what follows, then, assume that $\kappa \in M$ with (κ is a strongly inaccessible cardinal)^M.

The final basic property of $\mathrm{Col}(\omega, \kappa)^M$ that we need is the following.

Proposition 4. *($\mathrm{Col}(\omega, \kappa)$ satisfies the κ-chain condition)^M.*[8]

To prove this, we follow the approach of Kanamori [3, p. 127] and use the Δ-system lemma. The reader should be familiar with the case $\theta = \aleph_1$.

Lemma 3 (Δ-system lemma). *Let θ be an uncountable regular cardinal, and let $\mathcal{A}$ be a family of finite sets with $|\mathcal{A}| = \theta$. Then there exists $\mathcal{B} \subseteq \mathcal{A}$ with $|\mathcal{B}| = \theta$ and a finite set r such that for all $a, b \in \mathcal{B}$ with $a \neq b$, $a \cap b = r$.*

Proof. The proof is very similar to the case $\theta = \aleph_1$. See Kunen [1, p. 166] for details. □

Proof of Proposition 4. We carry out the proof inside M. Suppose $A \subseteq \mathrm{Col}(\omega, \kappa)$ with $|A| \geq \kappa$. Now apply the Δ-system lemma with $\theta = \kappa$ and $\mathcal{A} \subseteq \mathrm{dom}(A)$ to obtain $B \subseteq A$ with $|B| = \kappa$ and a finite set r such that for all $p, q \in B$ with $p \neq q$, $\mathrm{dom}(p) \cap \mathrm{dom}(q) = r$. Write $r = \{(\alpha_1, n_1), (\alpha_2, n_2), \ldots, (\alpha_k, n_k)\}$. Then for every $p \in B$, the image of $p\restriction_r$ is a subset of $\bigcup_{i=1}^{k} \alpha_i < \kappa$. So since $|B| = \kappa$, by the pigeonhole principle there exist $p, q \in B$ with $p\restriction_r = q\restriction_r$. Then p and q are compatible, so A is not an antichain. □

We are now ready to study the forcing extensions produced using $\mathrm{Col}(\omega, \kappa)^M$. In what follows, then, let G be a $\mathrm{Col}(\omega, \kappa)$-generic filter over M.

Proposition 2 then tells us that every ordinal less than κ is collapsed to a countable ordinal in $M[G]$, so $\kappa \leq \aleph_1^{M[G]}$. Our next observation is that κ is not itself collapsed to a countable ordinal in $M[G]$, from which it follows that $\kappa \geq \aleph_1^{M[G]}$. Strictly speaking, this second inequality will not be needed in the proof of Solovay's theorem, but we record it nonetheless since it follows easily from Proposition 4.

[8]A poset P satisfies the κ-chain condition iff all its antichains have cardinality less than κ. Here, an antichain of P is a set of pairwise incompatible members of P.

Corollary 1. $\kappa = \aleph_1^{M[G]}$.

Proof. Recall that if $P \in M$ is a forcing poset with (P satisfies the countable chain condition)^M, then P preserves cardinals $\geq \aleph_0$. By a very similar proof (using the fact that (κ is regular)^M), it follows from Proposition 4 that $\mathrm{Col}\,(\omega, \kappa)^M$ preserves cardinals $\geq \kappa$ [1, p. 291].

In particular, (κ is a cardinal)$^{M[G]}$. But $\kappa \neq \omega^{M[G]}$ by absoluteness, so $\kappa \geq \aleph_1^{M[G]}$. To complete the proof, by Proposition 2 every ordinal less than κ is collapsed to a countable ordinal in $M\,[G]$, so $\kappa \leq \aleph_1^{M[G]}$. □

The remaining results in this section concern intermediate extensions.

Notation. Let $x \in M\,[G]$. Denote by $M\,[x]$ the smallest subset of $M\,[G]$ such that $M \subseteq M\,[x]$, $x \in M\,[x]$ and $M\,[x]$ is a $\in$-model of ZFC.

The intermediate extensions we are concerned with are those of the form $M\,[f]$ for $f \in M\,[G]$ a sequence of ordinals. These will be important in the proof of Solovay's theorem when we consider ordinal-sequence-definability.

We begin by examining the behaviour of κ in these intermediate extensions. The following result should be contrasted with Corollary 1. All we will need from it later, however, is that $\left(2^{\aleph_0}\right)^{M[f]} < \kappa$.

Theorem 4. *Let $f \in M\,[G]$ with $f : \omega \to On^M$. Then (κ is a strongly inaccessible cardinal)$^{M[f]}$.*

To prove this, we will use the following lemma to relate our intermediate extension to a forcing extension.

Lemma 4. *Let $f \in M\,[G]$ with $f : \omega \to On^M$. Then there exists $\alpha < \kappa$ such that $M\,[f] \subseteq M\,[H]$, where $H = G \cap \mathrm{Col}\,(\omega, \alpha)^M$ is a $\mathrm{Col}\,(\omega, \alpha)$-generic filter over M.*

We follow the proof given in Kanamori [3, pp. 127–128]. The idea is to choose maximal antichains forcing the value of $f\,(n)$ for each $n \in \omega$ and then apply the κ-chain condition.

Proof. Let $\mathbf{f} \in M$ be a $\mathrm{Col}\,(\omega, \kappa)^M$-name whose interpretation relative to G is f.

For each $n \in \omega$, let A_n be a maximal antichain of $\mathrm{Col}\,(\omega, \kappa)^M$ such that for each $p \in A_n$ there exists $\beta \in On^M$ with $p \Vdash \mathbf{f}\,(n) = \beta$. Note that by the definability lemma we may assume $(A_n)_{n\in\omega} \in M$, since we avoided referencing f in the definition of $(A_n)_{n\in\omega}$.

Let $n \in \omega$. We claim that there exists a unique $p \in G \cap A_n$. To see this, observe that by the truth lemma, every $q \in \mathrm{Col}\,(\omega, \kappa)^M$ is compatible with some $r \in \mathrm{Col}\,(\omega, \kappa)^M$ satisfying $r \Vdash \mathbf{f}\,(n) = \beta$ for some $\beta \in On^M$. It follows that the set $\left\{q \in \mathrm{Col}\,(\omega, \kappa)^M : q \leq r \text{ for some } r \in A_n\right\}$ of predecessors of members of A_n is a dense subset of $\mathrm{Col}\,(\omega, \kappa)^M$ lying in M. So since G is $\mathrm{Col}\,(\omega, \kappa)$-generic over M and upward-closed, there exists $p \in G \cap A_n$, which must be unique since A_n is an antichain.

Let $A = \bigcup_{n\in\omega} A_n$. For each $p \in A$, let $\alpha_p = \bigcup_{(\beta,m)\in\mathrm{dom}(p)} \beta{+}1 < \kappa$, and let $\alpha = \bigcup_{p\in A} \alpha_p$. Then $\mathrm{dom}\,(p) \subseteq \alpha \times \omega$ for every $p \in A$ by construction. So letting $H = G \cap \mathrm{Col}\,(\omega, \alpha)^M$, we have $G \cap A_n = H \cap A_n$ for each $n \in \omega$. This allows us to define f in $M\,[H]$ by declaring that, for each $n \in \omega$, $p \Vdash f\,(n) = \mathbf{f}\,(n)$, where p is the unique member of $H \cap A_n$. Here we are using the definability lemma and the fact that $(A_n)_{n\in\omega} \in M$. Thus $f \in M[H]$ and so $M[f] \subseteq M[H]$.

To see that $\alpha < \kappa$, by Proposition 4, $(|A_n| < \kappa)^M$ for each $n \in \omega$, and hence $(|A| < \kappa)^M$. So by definition of α, it follows from the fact that $(\kappa \text{ is regular})^M$ that $\alpha < \kappa$.

It remains to show that H is a $\mathrm{Col}(\omega, \alpha)$-generic filter over M. That H is a filter of $\mathrm{Col}(\omega, \alpha)^M$ follows easily from the fact that G is a filter of $\mathrm{Col}(\omega, \kappa)^M$. To see that H is $\mathrm{Col}(\omega, \alpha)$-generic over M, observe that if $D \in M$ is a dense subset of $\mathrm{Col}(\omega, \alpha)^M$, then $D \times \left\{p \in \prod_{\alpha\le\beta<\kappa} [\omega \rightharpoonup \beta] : |p| < \aleph_0\right\}$ (viewed as a subset of $[\kappa \times \omega \rightharpoonup \kappa]$) is a dense subset of $\mathrm{Col}(\omega, \kappa)^M$ lying in M whose intersection with $\mathrm{Col}(\omega, \alpha)^M$ is D. Now apply $\mathrm{Col}(\omega, \kappa)$-genericity of G over M. □

The usefulness of this result in the proof of Theorem 4 comes from the fact that if $\alpha < \kappa$ then $(|\mathrm{Col}(\omega, \alpha)| < \kappa)^M$. We use ideas from Jech [2, p. 226].

Proof of Theorem 4. First note that by Corollary 1, $(\kappa \text{ is a regular cardinal})^{M[G]}$. But this is a statement about the non-existence of certain injections in $M[G]$, so there cannot be such injections in $M[f]$ either, so $(\kappa \text{ is a regular cardinal})^{M[f]}$.

To see that $(\kappa \text{ is a strong limit})^{M[f]}$, choose $\alpha < \kappa$ as in Lemma 4 and let $H = G \cap \mathrm{Col}(\omega, \alpha)$. Since $M[f] \subseteq M[H]$, by a similar argument to the one in the above paragraph it is sufficient to prove that $(\kappa \text{ is a strong limit})^{M[H]}$. So let $\nu < \kappa$ with $(\nu \text{ is a cardinal})^{M[H]}$.

Then every subset $X \subseteq \nu$ with $X \in M[H]$ has a corresponding $\mathrm{Col}(\omega, \alpha)$-name $\mathbf{X} \in M$ of the form $\mathbf{X} = \{(p_x, \check{x}) : x \in \nu\}$. This defines an injection

$$\mathcal{P}(\nu)^{M[H]} \to \left\{\{(p_x, \check{x}) : x \in \nu\} \in M^{\mathrm{Col}(\omega,\alpha)}\right\}^M .$$

So $(2^\nu)^{M[H]} \le (|\mathrm{Col}(\omega, \alpha)|^\nu)^M$ (1).

But (assuming without loss of generality that $\alpha \ge \omega$)

$$\left(|\mathrm{Col}(\omega, \alpha)| \le \prod_{\beta<\alpha} |\beta|^{\aleph_0} \le |\alpha|^{\aleph_0\cdot|\alpha|} = |\alpha|^{|\alpha|} = 2^{|\alpha|}\right)^M .$$

So since $\alpha < \kappa$ and $(\kappa \text{ is a strong limit})^M$, $\left(|\mathrm{Col}(\omega, \alpha)|^\nu \le 2^{|\alpha|\cdot\nu} < \kappa\right)^M$ (2).

Together the inequalities (1) and (2) yield $(2^\nu)^{M[H]} < \kappa$. □

The final crucial result that we will need later is the following.

Theorem 5. *Let $\phi(x)$ be a formula in the language of set theory. Then there is a formula $\widetilde{\phi}(x)$ in the language of set theory such that for any $f \in M[G]$ with $f : \omega \to On^M$,*

$$\phi(f)^{M[G]} \text{ iff } \widetilde{\phi}(f)^{M[f]} .$$

We follow the proof given in Kanamori [3, p. 140]. The ingredients are Proposition 3 and the following lemma, which tells us how sequences of ordinals can be "absorbed" into M.

Lemma 5 (factor lemma). *Let $f \in M[G]$ with $f : \omega \to On^M$. Then there is a* $\mathrm{Col}(\omega, \kappa)$*-generic filter H over $M[f]$ such that $M[G] = M[f][H]$.*

For the proof of the factor lemma, we refer the reader to Kanamori [3, pp. 129–131] or Jech [2, pp. 516–518]. A sketch of Kanamori's proof can be found in Appendix A.

Proof of Theorem 5. Let $f \in M[G]$ with $f : \omega \to On^M$. Then by the factor lemma there is a $\mathrm{Col}(\omega, \kappa)$-generic filter H over $M[f]$ such that $M[G] = M[f][H]$. Now by Proposition 3, either $\emptyset \Vdash_{M[f]} \phi(\check{f})$ or $\emptyset \Vdash_{M[f]} \neg\phi(\check{f})$. It follows that $\phi(f)^{M[f][H]}$ iff $\emptyset \Vdash_{M[f]} \phi(\check{f})$.

Now by the definability lemma for forcing extensions, there is a single formula $\widetilde{\phi}(x)$ in the language of set theory such that for any $f \in M[G]$ with $f : \omega \to On^M$, $\widetilde{\phi}(f)^{M[f]}$ iff $\emptyset \Vdash_{M[f]} \phi(\check{f})$. Then for any such f, $\phi(f)^{M[G]}$ iff $\widetilde{\phi}(f)^{M[f]}$, as required. □

Remark. Corollary 1, Theorem 4 and Theorem 5 all use the fact that $(\kappa \text{ is regular})^M$. Only Theorem 4 uses the fact that $(\kappa \text{ is a strong limit})^M$.

2.3 Borel sets and Borel codes

Now that we have studied the model-theoretic constructions to be used in the proof of Solovay's theorem, we turn to the measure-theoretic side of the proof.

We will need to apply some of the results of this section in a model of $\mathsf{ZF} + \mathsf{DC}$ in which AC need not hold, so in this section, we work in $\mathsf{ZF} + \mathsf{DC}$ rather than ZFC.

Definition. The *Borel algebra* $\mathcal{B}$ of $\mathbb{R}$ is the smallest subset of $\mathcal{P}(\mathbb{R})$ such that:

(a) if $A \subseteq \mathbb{R}$ is open, then $A \in \mathcal{B}$;

(b) (*complementation*) if $A \in \mathcal{B}$, then $\mathbb{R} \setminus A \in \mathcal{B}$;

(c) (*countable union*) if $(A_i)_{i \in I}$ is a countable collection of members of $\mathcal{B}$, then $\bigcup_{i \in I} A_i \in \mathcal{B}$.

A set $A \subseteq \mathbb{R}$ is a *Borel set* iff $A \in \mathcal{B}$.
A set $A \subseteq \mathbb{R}$ is a *null set* iff its outer Lebesgue measure $\mu^*(A) = 0$.

The connection between Borel sets and Lebesgue measurability is given by the following result [2, p. 147], which we will use in the proof of Solovay's theorem to demonstrate Lebesgue measurability. The proof can be found in any elementary text on measure theory.

Fact 1. *A set $A \subseteq \mathbb{R}$ is Lebesgue measurable iff there is a Borel set B such that $A \triangle B$*[9] *is null; a set of reals is null iff it is a subset of a null Borel set.*

Although it will not be needed in the proof of Solovay's theorem, it is instructive to have the following more explicit description of $\mathcal{B}$ known as the *Borel hierarchy*. The following definition and proposition are based on Jech [2, p. 140].

Definition. For each $\alpha \in \omega_1 \setminus \{0\}$, define $\mathbf{\Sigma}^0_\alpha, \mathbf{\Pi}^0_\alpha \subseteq \mathcal{P}(\mathbb{R})$ recursively by:

- $\mathbf{\Sigma}^0_1 := \{A \subseteq \mathbb{R} : A \text{ is open}\}$;

[9] $A \triangle B$ denotes the symmetric difference $(A \setminus B) \cup (B \setminus A)$.

- $\mathbf{\Pi}^0_1 := \{A \subseteq \mathbb{R} : A \text{ is closed}\}$;
- for $\alpha > 1$, $\mathbf{\Sigma}^0_\alpha := \left\{\bigcup_{i=0}^\infty A_i : (\forall i \in \omega)\,(\exists \beta < \alpha)\,\left(A_i \in \mathbf{\Pi}^0_\beta\right)\right\}$;
- for $\alpha > 1$, $\mathbf{\Pi}^0_\alpha := \{\mathbb{R} \setminus A : A \in \mathbf{\Sigma}^0_\alpha\}$
  $= \left\{\bigcap_{i=0}^\infty A_i : (\forall i \in \omega)\,(\exists \beta < \alpha)\,\left(A_i \in \mathbf{\Sigma}^0_\beta\right)\right\}$.

Proposition 5. $\mathcal{B} = \bigcup_{\alpha<\omega_1} \mathbf{\Sigma}^0_\alpha = \bigcup_{\alpha<\omega_1} \mathbf{\Pi}^0_\alpha$.

Proof. Firstly, it is clear by induction on α that $\bigcup_{\alpha<\omega_1} \mathbf{\Sigma}^0_\alpha, \bigcup_{\alpha<\omega_1} \mathbf{\Pi}^0_\alpha \subseteq \mathcal{B}$.

Next, observe that every open subset of $\mathbb{R}$ contains a closed interval with rational endpoints about each of its points. Therefore every open subset of $\mathbb{R}$ can be written as a countable union of closed sets, and hence $\mathbf{\Sigma}^0_1 \subseteq \mathbf{\Sigma}^0_2$. From this it follows easily that for any $\alpha, \beta \in \omega_1 \setminus \{0\}$ with $\alpha < \beta$, $\mathbf{\Sigma}^0_\alpha \subseteq \mathbf{\Sigma}^0_\beta$, $\mathbf{\Sigma}^0_\alpha \subseteq \mathbf{\Pi}^0_\beta$, $\mathbf{\Pi}^0_\alpha \subseteq \mathbf{\Pi}^0_\beta$ and $\mathbf{\Pi}^0_\alpha \subseteq \mathbf{\Sigma}^0_\beta$. Hence $\bigcup_{\alpha<\omega_1} \mathbf{\Sigma}^0_\alpha = \bigcup_{\alpha<\omega_1} \mathbf{\Pi}^0_\alpha$.

It follows from this that $\bigcup_{\alpha<\omega_1} \mathbf{\Sigma}^0_\alpha$ is closed under complementation and countable union, and hence $\mathcal{B} \subseteq \bigcup_{\alpha<\omega_1} \mathbf{\Sigma}^0_\alpha$. □

Thus every Borel set can be obtained from open subsets of $\mathbb{R}$ through complementation and countable union in fewer than ω_1 steps. The idea of a Borel code is to describe, using members of ${}^\omega\omega$, this procedure for obtaining a Borel set. This will be useful in the proof of Solovay's theorem when we concern ourselves with absoluteness and ordinal-sequence-definability. We use ideas from Jech [2, pp. 504–505].

Given a Borel set described either as an open set, or as the complement of another set, or as the union of a countable sequence of other sets, we seek to construct a code $c \in {}^\omega\omega$ for this description recursively. We will use $c\,(0)$ to declare which of the three types of description is being used, and then use the rest of c either to describe an open set, or another set, or a countable sequence of other sets, respectively. To help us describe open sets, we fix a recursive[10] enumeration $I_1, I_2, \ldots$ of the open intervals with rational endpoints. To help us describe a countable sequence of other sets, we fix a recursive bijection $\Gamma : \omega \times \omega \to \omega$. The recursiveness is to ensure absoluteness below.

Write L for the "left-shift" function ${}^\omega\omega \to {}^\omega\omega$ defined by $L\,(c_0, c_1, \ldots) = (c_1, c_2, \ldots)$, and for $i \in \omega$, write S_i for the "ith subsequence" function ${}^\omega\omega \to {}^\omega\omega$ induced by Γ, defined by $S_i\,(c_0, c_1, \ldots) = \left(c_{\Gamma(i,0)}, c_{\Gamma(i,1)}, \ldots\right)$.

Definition. The set BC of *Borel codes* is the smallest subset of ${}^\omega\omega$ with the following properties. At the same time we define the *interpretation* A_c of $c \in \text{BC}$ recursively.

(a) If $c\,(0) > 1$, then c is a Borel code and $A_c = \bigcup \{I_n : n \in \omega \setminus \{0\} \text{ and } c\,(n) = 1\}$;

(b) if $c\,(0) = 0$ and $L\,(c)$ is a Borel code, then c is a Borel code and $A_c = \mathbb{R} \setminus A_{L(c)}$;

(c) if $c\,(0) = 1$ and $S_i\,(L\,(c))$ is a Borel code for every $i \in \omega$, then c is a Borel code and $A_c = \bigcup_{i=0}^\infty A_{S_i(L(c))}$.

For $c \in \text{BC}$, if $c\,(0) > 1$ then we say c is an *open code*, and if $c\,(0) = 0$ and $L\,(c)$ is an open code, then we say c is a *closed code*.

[10] Here, by "recursive" we mean "recursive when viewed as a function $\omega \setminus \{0\} \to \mathbb{Q}^2$".

Proposition 6.

1. $\{A_c : c \in \mathrm{BC}$ *and c is an open code*$\} = \{O \subseteq \mathbb{R} : O$ *is open*$\}$;
2. $\{A_c : c \in \mathrm{BC}$ *and c is a closed code*$\} = \{C \subseteq \mathbb{R} : C$ *is closed*$\}$;
3. $\{A_c : c \in \mathrm{BC}\} = \mathcal{B}$.

Proof.

1. It is sufficient to prove that every open subset of $\mathbb{R}$ can be written as a countable union of intervals with rational endpoints. But this is clear, since every open subset of $\mathbb{R}$ contains an open interval with rational endpoints about each of its points, and there are countably many such intervals.
2. This follows immediately from part 1.
3. Given part 1, it is clear that the definition of $\{A_c : c \in \mathrm{BC}\}$ precisely mirrors that of $\mathcal{B}$. □

The power of Borel codes comes from their absoluteness properties.

Theorem 6. *Suppose M is a transitive* $\in$*-model of* $\mathsf{ZF} + \mathsf{DC}$. *Then:*

1. $\mathrm{BC}^M = \mathrm{BC} \cap M$;
2. *if* $c \in \mathrm{BC}^M$, *then* $A_c^M = A_c \cap M$;
3. *if* $c \in \mathrm{BC}^M$ *is an open code or a closed code, then* $\mu(A_c)^M = \mu(A_c)$;[11]
4. *if* $c \in \mathrm{BC}^M$, *then* (A_c *is null*)^M *iff* A_c *is null.*

For the proof of parts 1 and 2, we ignore the technicalities of how recursive definitions are formalised and assume that the above definitions retain their form when relativised to M. A rigorous proof can be carried out via Mostowski's absoluteness theorem (see Jech [2, pp. 483–484, 505–506]).

For the proof of parts 3 and 4, we follow the approach of Jech [2, p. 512] and appeal to the following basic property of Lebesgue measure for part 4.

Fact 2. *A set* $B \in \mathcal{B}$ *is null iff for every* $n \in \mathbb{N}$,[12] *there exists an open set* $O \subseteq \mathbb{R}$ *with* $O \supseteq B$ *and* $\mu(O) < \frac{1}{n}$.

A set $B \in \mathcal{B}$ *is not null iff for every* $n \in \mathbb{N}$, *there exists a closed set* $C \subseteq \mathbb{R}$ *with* $C \subseteq B$ *and* $\mu(C) > \frac{1}{n}$.

Proof of Theorem 6.

1. First note that both the sequence $(I_k)_{k\in\mathbb{N}}$ and the function Γ lie in M since they are recursive, so the definition of BC makes sense in M. By our simplifying assumption, it follows by induction on $c \in \mathrm{BC}$ that $c \in M$ iff $c \in \mathrm{BC}^M$.

[11] Of course by $\mu(A_c)^M$ we mean $\mu^M\left(A_c^M\right)$.

[12] By $\mathbb{N}$ we mean $\omega \setminus \{0\}$.

2. Since $\mathcal{P}(\omega)^M = \mathcal{P}(\omega) \cap M$, we may assume that $\mathbb{R}$ is constructed in such a way that $\mathbb{R}^M = \mathbb{R} \cap M$. By our simplifying assumption again, it follows by induction on $c \in \mathrm{BC}$ that if $c \in M$ then $A_c^M = A_c \cap M$.

3. Let $c \in \mathrm{BC}^M$ be an open code. Then $A_c = \bigcup_{i\in\omega} I_{k_i}$ for some sequence $(k_i)_{i\in\omega} \in M$ of members of $\mathbb{N}$. For each $i \in \omega$, let $J_i = I_{k_i} \setminus \left(I_{k_0} \cup I_{k_1} \cup \cdots \cup I_{k_{i-1}}\right)$, so that $(J_i)_{i\in\omega}$ is a sequence of disjoint sets with $\bigcup_{i\in\omega} J_i = \bigcup_{i\in\omega} I_{k_i} = A_c$. Then for each $i \in \omega$, J_i can be written as a disjoint union of intervals with rational endpoints, so $\mu(J_i)^M = \mu(J_i)$. Hence by countable additivity of Lebesgue measure together with the same applied in M, $\mu(A_c)^M = \sum_{i\in\omega} \mu(J_i)^M = \sum_{i\in\omega} \mu(J_i) = \mu(A_c)$.

   The case in which $c \in \mathrm{BC}^M$ is a closed code is similar.

4. Let $c \in \mathrm{BC}^M$. If $(A_c \text{ is null})^M$, then by the first half Fact 2 and part 1 of Proposition 6 applied in M, for every $n \in \mathbb{N}$ there exists an open code $d_n \in \mathrm{BC}^M$ with $A_{d_n}^M \supseteq A_c^M$ and $\mu(A_{d_n})^M < \frac{1}{n}$. For each $n \in \mathbb{N}$, by parts 1, 2 and 3 it follows that d_n is an open code, $A_{d_n} \supseteq A_c$ and $\mu(A_{d_n}) < \frac{1}{n}$. So by Fact 2, A_c is null.

   Conversely, if $(A_c \text{ is not null})^M$, then by a similar argument using the second half of Fact 2, it follows that A_c is not null. □

Remark. Part 4 of this theorem seems to say that being null is absolute for a transitive $\in$-model M of $\mathsf{ZF} + \mathsf{DC}$. Indeed, using parts 2 and 4 of this theorem, we can deduce the following by applying part 3 of Proposition 6 in M: if $B \in \mathcal{B}^M$ then there exists $B' \in \mathcal{B}$ such that $B \subseteq B'$ and $(B' \text{ is null})^M$ iff B is null. However, one cannot conclude from this that if $B \in \mathcal{B}^M$ then $(B \text{ is null})^M$ iff B is null. Indeed, if M is countable then $\mathbb{R}^M = \mathbb{R} \cap M$ is countable and hence null, but clearly $(\mathbb{R} \text{ is not null})^M$. So in this sense, being null is not absolute for M.

2.4 Random reals and Solovay sets of reals

With these basic properties of Borel sets and Borel codes in place, we are ready to turn to the concepts at the heart of the proof of Solovay's theorem. We use ideas from Jech [2, pp. 513–515].

The concept of a "random" real arises from forcing with a poset based on the Borel algebra $\mathcal{B}$ ordered by $\subseteq$. Rather than using $\mathcal{B}$ itself, we would like, intuitively speaking, to replace the notion of "empty" by the notion of "null".[13] Now given two sets A and B, $A = B$ iff $A \triangle B$ is empty, and $A \subseteq B$ iff $A \setminus B$ is empty. This motivates the following definition.

Definition. Define the relation $\sim$ on $\mathcal{B}$ by

$$A \sim B \text{ iff } A \triangle B \text{ is null.}$$

It follows from the basic properties of Lebesgue measure (using additivity to prove transitivity) that $\sim$ is an equivalence relation on $\mathcal{B}$. So we may write $[B]$ for the equivalence class of $B \in \mathcal{B}$.

[13] Note that since all Borel sets are Lebesgue measurable, $A \in \mathcal{B}$ is null iff the Lebesgue measure $\mu(A) = 0$.

Define the relation $\lesssim$ on $\mathcal{B}/\sim$ by

$$[A] \lesssim [B] \text{ iff } A \setminus B \text{ is null.}$$

It follows again from the basic properties of Lebesgue measure that $\lesssim$ is well-defined and a partial order on $\mathcal{B}$.

Define the forcing poset $\widetilde{\mathcal{B}}$ by $(\mathcal{B}/\sim) \setminus \{[\emptyset]\}$, ordered by $\lesssim$.

Remarks.

1. Since we have excluded the equivalence class $[\emptyset]$ of null sets, it follows easily that $\widetilde{\mathcal{B}}$ is separative.

2. By Fact 1, we could have replaced $\mathcal{B}$ in this definition by $\mathcal{M}$, where $\mathcal{M} = \{A \subseteq \mathbb{R} : A \text{ is Lebesgue measurable}\}$, and then the map $\widetilde{\mathcal{B}} \to \widetilde{\mathcal{M}}$ given by $[B] \mapsto [B]$ would be well-defined and an order-isomorphism.

3. The remaining results of the section make frequent implicit use of the absoluteness properties given by Theorem 6. For example, it follows from parts 2 and 4 that for $c \in \mathrm{BC}^M$, $[A_c]^M = \left\{A_d \cap M : d \in \mathrm{BC}^M, A_d \in [A_c]\right\}$, and so for $c, d \in \mathrm{BC}^M$, $[A_c]^M = [A_d]^M$ iff $[A_c] = [A_d]$.

We begin by illustrating a basic property of $\widetilde{\mathcal{B}}$ that will be useful later.

Lemma 6. *$\widetilde{\mathcal{B}}$ satisfies the countable chain condition.*

Proof. Suppose $\mathcal{A} \subseteq \widetilde{\mathcal{B}}$ is uncountable, and suppose for contradiction that $\mathcal{A}$ is an antichain, meaning that $A \cap B$ is null for all $[A], [B] \in \mathcal{A}$.

Let $[A] \in \mathcal{A}$. For each $m \in \mathbb{Z}$, let $\mu_m(A) = \mu(A \cap [m, m+1))$. Then by countable additivity of Lebesgue measure, $\mu(A) = \sum_{m \in \mathbb{Z}} \mu_m(A)$. So since $\mu(A) > 0$, there exists $m_{[A]} \in \mathbb{Z}$ with $\mu_{m_{[A]}}(A) > 0$. Note that this makes sense because if $A, B \in \mathcal{B}$ with $[A] = [B]$, then $\mu(A) = \mu(B)$.

Then since $\mu_{m_{[A]}}(A) > 0$ for all $[A] \in \mathcal{A}$, there exists $n \in \mathbb{N}$ and an uncountable subset $\mathcal{C} \subseteq \mathcal{A}$ such that $\mu_{m_{[A]}}(A) > \frac{1}{n}$ for all $[A] \in \mathcal{C}$. Then there exists $m \in \mathbb{Z}$ and a further uncountable subset $\mathcal{D} \subseteq \mathcal{C}$ such that $m_{[A]} = m$ for all $[A] \in \mathcal{D}$. Let $([A_i])_{i \in \omega}$ be a sequence of members of $\mathcal{D}$.

Now define $B_i = A_i \setminus (A_0 \cup A_1 \cup \cdots \cup A_{i-1})$ for each $i \in \omega$. It follows from the fact that $\mathcal{A}$ is an antichain that $[A_i] = [B_i]$ for all $i \in \omega$. But $(B_i)_{i \in \omega}$ is a sequence of pairwise disjoint sets by construction. Therefore by monotonicity and countable additivity of Lebesgue measure,

$$\mu([m, m+1)) \geq \mu_m\left(\bigcup_{i \in \omega} B_i\right) = \sum_{i \in \omega} \mu_m(B_i) = \sum_{i \in \omega} \mu_m(A_i) \geq \sum_{i \in \omega} \frac{1}{n} = \infty.$$

Contradiction. □

We now consider what happens when we force with $\widetilde{\mathcal{B}}$. Assume, then, that M is a countable transitive $\in$-model of ZFC.

Roughly speaking, a $\widetilde{\mathcal{B}}$-generic filter over M is the collection of M-Borel neighbourhoods of a generic real. We shall now use this argument to show that forcing with $\widetilde{\mathcal{B}}$ amounts to adjoining a real. It is the reals that can be adjoined in this fashion that we will call random (over M).

The following proposition and definition are based on Jech [2, p. 513].

Proposition 7. *Let G be a $\widetilde{\mathcal{B}}$-generic filter over M. Then there exists a unique $x_G \in \mathbb{R}$ such that for all $c \in \mathrm{BC}^M$,*

$$x_G \in A_c \text{ iff } [A_c]^M \in G. \tag{$*$}$$

Moreover, $M[G] = M[x_G]$.

Definition. We say $x \in \mathbb{R}^{M[G]}$ is *random over M*, and write $x \in R(M)$, iff there is a $\widetilde{\mathcal{B}}$-generic filter G over M such that $x = x_G$.

Proof of Proposition 7. We begin by defining x_G in $M[G]$. For each $n \in \mathbb{N}$, let

$$D_n = \left\{ [B]^M \in \widetilde{\mathcal{B}}^M : B \subseteq (a,b)^M \text{ for some } a, b \in \mathbb{Q} \text{ with } a < b < a + \tfrac{1}{n} \right\},$$

a dense subset of $\widetilde{\mathcal{B}}^M$ lying in M. For each $n \in \mathbb{N}$, choose $[B_n]^M \in D_n \cap G$ with $B_n \subseteq (a_n, b_n)^M$, where $a_n, b_n \in \mathbb{Q}$ and $a_n < b_n < a_n + \frac{1}{n}$. Then since G is upward-closed, $[(a_n, b_n)]^M \in G$ for each $n \in \mathbb{N}$, and since G is directed, $a_m < b_n$ for all $m, n \in \mathbb{N}$, else $B_n \cap B_m$ is null. It follows easily that $\sup\{a_n : n \in \mathbb{N}\}$ and $\inf\{b_n : n \in \mathbb{N}\}$ exist and are equal. Take x_G to be their common value. Note that since G is directed, the value of x_G is independent of the particular choices made.

Observe that for all $a, b \in \mathbb{Q}$ with $a < x_G < b$, there exists $n \in \mathbb{N}$ with $(a_n, b_n) \subseteq (a, b)$. By directedness and upward-closedness of G respectively, it follows that for all $c \in \mathrm{BC}^M$:

- if there exists $a, b \in \mathbb{Q}$ with $a < x_G < b$ such that $A_c \cap (a,b)$ is null, then $[A_c]^M \notin G$;
- if there exists $a, b \in \mathbb{Q}$ with $a < x_G < b$ such that $([(a,b)] \lesssim [A_c])^M$, then $[A_c]^M \in G$.

This allows us to define G in $M[x_G]$.

Thus $x_G \in M[G]$ and $G \in M[x_G]$, so $M[G] = M[x_G]$. Since $\widetilde{\mathcal{B}}^M$ is separative, it follows that $x_G \notin M$ and hence, crucially, $x_G \notin \mathbb{Q}$.

Next, the uniqueness property of x_G follows easily, for if $y \in \mathbb{R} \setminus \{x_G\}$, then there exists $n \in \mathbb{N}$ with $y \notin (a_n, b_n)$. Since this is simply a single interval with rational endpoints, there exists $c \in \mathrm{BC}^M$ with $A_c = (a_n, b_n)$. Then $[A_c]^M \in G$ but $y \notin A_c$, so y does not have property $(*)$.

To complete the proof, we prove by induction on $c \in \mathrm{BC}^M$ that x_G has property $(*)$.

(a) If $c(0) > 1$, then A_c is an open subset of $\mathbb{R}$. Since countable unions will be covered in part (c), we may assume $A_c = (a,b)$ for some $a, b \in \mathbb{Q}$. But then since $x_G \notin \mathbb{Q}$, $x_G \in (a,b)$ iff $(a_n, b_n) \subseteq (a,b)$ for some $n \in \mathbb{N}$ iff $[(a,b)]^M \in G$.

(b) If $c(0) = 0$, then $L(c) \in \mathrm{BC}^M$ and $A_c = \mathbb{R} \setminus A_{L(c)}$, and by the induction hypothesis $x_G \in A_{L(c)}$ iff $\left[A_{L(c)}\right]^M \in G$. Let

$$D = \left\{ [B]^M \in \widetilde{\mathcal{B}}^M : ([B] \lesssim [A_c])^M \text{ or } \left([B] \lesssim \left[A_{L(c)}\right]\right)^M \right\},$$

a dense subset of $\widetilde{\mathcal{B}}^M$ lying in M. Then $G \cap D$ is non-empty, so exactly one of $[A_c]^M$ and $\left[A_{L(c)}\right]^M$ lies in G. So $x_G \in A_c$ iff $x_G \notin A_{L(c)}$ iff $\left[A_{L(c)}\right]^M \notin G$ iff $[A_c]^M \in G$.

(c) If $c(0) = 1$, then $S_i(L(c)) \in \mathrm{BC}^M$ for every $i \in \omega$ and $A_c = \bigcup_{i=0}^{\infty} A_{S_i(L(c))}$, and by the induction hypothesis, $x_G \in A_{S_i(L(c))}$ iff $\left[A_{S_i(L(c))}\right]^M \in G$ for every $i \in \omega$. Let

$$D = \left\{[B]^M \in \widetilde{\mathcal{B}}^M : \left([B] \lesssim \left[A_{S_i(L(c))}\right]\right)^M \text{ for some } i \in \omega \text{ or } \left([B] \lesssim [\mathbb{R} \setminus A_c]\right)^M\right\},$$

a dense subset of $\widetilde{\mathcal{B}}^M$ lying in M. Then $G \cap D$ is non-empty, so $[A_c]^M \in G$ iff $\left[A_{S_i(L(c))}\right]^M \in G$ for some $i \in \omega$. So $x_G \in A_c$ iff $x_G \in A_{S_i(L(c))}$ for some $i \in \omega$ iff $\left[A_{S_i(L(c))}\right]^M \in G$ for some $i \in \omega$ iff $[A_c]^M \in G$. □

From Proposition 7 we can deduce an important characterisation of random reals. The following proposition is based on Jech [2, pp. 514–515] and Kanamori [3, p. 139].

Proposition 8. *$R(M) = \mathbb{R} \setminus \bigcup \left\{A_c : c \in \mathrm{BC}^M \text{ and } A_c \text{ is null}\right\}$.*

Proof. For the forward inclusion, suppose $r \in R(M)$. Then $r = x_G$ for some $\widetilde{\mathcal{B}}$-generic filter G over M. Now if $c \in \mathrm{BC}^M$ and A_c is null, then $[A_c]^M = [\emptyset]^M$ does not even lie in $\widetilde{\mathcal{B}}^M$, let alone G, so by property $(*)$, $x_G \notin A_c$. Hence $r \in \mathbb{R} \setminus \bigcup \left\{A_c : c \in \mathrm{BC}^M \text{ and } A_c \text{ is null}\right\}$.

For the reverse inclusion, suppose $r \in \mathbb{R} \setminus \bigcup \left\{A_c : c \in \mathrm{BC}^M \text{ and } A_c \text{ is null}\right\}$. Then let

$$G = \left\{[A_c]^M : c \in \mathrm{BC}^M \text{ and } r \in A_c\right\}.$$

Note that $G \subseteq \widetilde{\mathcal{B}}^M$ by definition of r. It is sufficient to prove that G is a $\widetilde{\mathcal{B}}$-generic filter over M, for it then follows by uniqueness in Proposition 7 that $r = x_G$ and hence $r \in R(M)$.

To see this, it is clear that G is a filter. For genericity, let $D \in M$ be a dense subset of $\widetilde{\mathcal{B}}$. Then let $\mathcal{A} \in M$ be a maximal antichain of $\widetilde{\mathcal{B}}^M$ such that for all $p \in \mathcal{A}$ there exists $q \in D$ with $(p \lesssim q)^M$. Then by applying Lemma 6 in M, $(\mathcal{A} \text{ is countable})^M$. So we may let $d \in \mathrm{BC}^M$ be the Borel code describing the set $\mathbb{R} \setminus \bigcup \{A_c : c \in C\}$, where C is defined by $\mathcal{A} = \left\{[A_c]^M : c \in C\right\}$.

Then A_d is null, else by density of D we would be able to find $p \in D$ with $(p \lesssim [A_d])^M$, which we could then add to $\mathcal{A}$ to obtain a larger antichain. It follows by definition of r that $r \notin A_d$. So by definition of d, there exists $c \in C$ with $r \in A_c$. Then $[A_c]^M \in G \cap \mathcal{A}$. So by definition of $\mathcal{A}$, there exists $q \in D$ with $([A_c] \lesssim q)^M$. Then $q \in G \cap D$ since G is upward-closed. Thus G is $\widetilde{\mathcal{B}}$-generic over M. □

The bridge between Lebesgue measurability and definability in the proof of Solovay's theorem comes from the notion of a Solovay set of reals. The following definition and proposition are based on Jech [2, p. 515].

Definition. Let $S \subseteq \mathbb{R}$. We say S is *Solovay over M* iff there exists a formula $\phi(x_1, x_2, \ldots, x_n, y)$ in the language of set theory and $a_1, a_2, \ldots, a_n \in M$ such that for all $r \in \mathbb{R}$,

$$r \in S \text{ iff } \phi(a_1, a_2, \ldots, a_n, r)^{M[r]}.$$

The connection between random reals and Solovay sets of reals is given by the following result.

Proposition 9. *Let $S \subseteq \mathbb{R}$ be Solovay over M. Then there exists $d \in \mathrm{BC}^M$ such that for all $r \in R(M)$,*

$$r \in S \text{ iff } r \in A_d.$$

The proof uses ideas from Jech [2, p. 515] and Kanamori [3, p. 140].

Proof. Recall the canonical $\widetilde{\mathcal{B}}^M$-name $\dot{G} := \left\{(p, \check{p}) : p \in \widetilde{\mathcal{B}}^M\right\}$ for a $\widetilde{\mathcal{B}}$-generic filter over M. Now by Proposition 7, if G is a $\widetilde{\mathcal{B}}$-generic filter over M, then $x_G = \bigcap \left\{A_c^{M[G]} : [A_c]^M \in G\right\}$. Via the proof of the forcing theorem, this allows us to construct a canonical $\widetilde{\mathcal{B}}^M$-name $\dot{x}$ for a random real, so that if G is a $\widetilde{\mathcal{B}}$-generic filter over M then the interpretation of $\dot{x}$ relative to G is x_G.

Since S is Solovay over M, there exists a formula $\phi(x_1, x_2, \ldots, x_n, y)$ and $a_1, a_2, \ldots, a_n \in M$ such that for all $r \in \mathbb{R}$, $r \in S$ iff $\phi(a_1, a_2, \ldots, a_n, r)^{M[r]}$. We now use an argument involving a maximal antichain that is very similar to the one in the proof of Lemma 4.

Let $\mathcal{A} \in M$ be a maximal antichain of $\widetilde{\mathcal{B}}^M$ such that for all $p \in \mathcal{A}$, either $p \Vdash \phi(\check{a}_1, \check{a}_2, \ldots, \check{a}_n, \dot{x})$ or $p \Vdash \neg\phi(\check{a}_1, \check{a}_2, \ldots, \check{a}_n, \dot{x})$. Partition $\mathcal{A}$ as $\mathcal{A}_+ \cup \mathcal{A}_-$ according to the two cases. Just as in the proof of Lemma 4, it follows by considering the set of $\lesssim$-predecessors of members of $\mathcal{A}$ that for any $\widetilde{\mathcal{B}}$-generic filter G over M, there is a unique $p_G \in G \cap \mathcal{A}$.

Now by applying Lemma 6 in M, $(\mathcal{A} \text{ is countable})^M$, so we may let $d \in \mathrm{BC}^M$ be the Borel code describing the countable union $\bigcup\{A_c : c \in C_+\}$, where C_+ is defined by $\mathcal{A}_+ = \left\{[A_c]^M : c \in C_+\right\}$. Then $(p \lesssim [A_d])^M$ for all $p \in \mathcal{A}_+$, and $(p \cap A_d \text{ is null})^M$ for all $p \in \mathcal{A}_-$ by countable additivity of Lebesgue measure.

We claim that d is as required. To see this, let $r \in R(M)$. Then $r = x_G$ for some $\widetilde{\mathcal{B}}$-generic filter G over M. So

$$\begin{aligned}
x_G \in S &\text{ iff } \phi(a_1, a_2, \ldots, a_n, x_G)^{M[x_G]} \\
&\text{ iff } \phi(a_1, a_2, \ldots, a_n, x_G)^{M[G]} \text{ (since } M[G] = M[x_G] \text{ by Proposition 7)} \\
&\text{ iff } p \Vdash \phi(\check{a}_1, \check{a}_2, \ldots, \check{a}_n, \dot{x}) \text{ for some } p \in G \text{ (by the truth lemma)} \\
&\text{ iff } p_G \in G \cap \mathcal{A}_+ \text{ (else } p_G \in G \cap \mathcal{A}_-) \\
&\text{ iff } [A_d]^M \in G \text{ (since } G \text{ is upward-closed)} \\
&\text{ iff } x_G \in A_d \text{ (by property } (*) \text{ from Proposition 7).}
\end{aligned}$$

In other words, $r \in S$ iff $r \in A_d$. □

All we will need from this section in the proof of Solovay's theorem is the following result, which brings together Propositions 8 and 9.

Corollary 2. *Let $S \subseteq \mathbb{R}$ be Solovay over M. Then there exists $d \in \mathrm{BC}^M$ such that*

$$S \triangle A_d \subseteq \bigcup \left\{A_c : c \in \mathrm{BC}^M \text{ and } A_c \text{ is null}\right\}.$$

Proof. By Proposition 9, there exists $d \in \mathrm{BC}^M$ such that for all $r \in R(M)$, $r \in S$ iff $r \in A_d$. In other words, $S \triangle A_d \subseteq \mathbb{R} \setminus R(M)$. Now apply Proposition 8. □

2.5 Proof of Solovay's theorem

With all the preliminary results in place, we can at last proceed to the proof of Solovay's theorem. We encourage the reader to review the outline given in the introduction to this chapter and the statements of Lemma 2, Fact 1, Corollaries 1 and 2 and Theorems 3–6. The proof is based on Jech [2, pp. 520–521].

Proof of Theorem 2. Let M be a countable transitive $\in$-model of $\mathsf{ZFC}+\mathsf{I}$. Let $\kappa \in M$ with $(\kappa \text{ is a strongly inaccessible cardinal})^M$, and let G be $\mathrm{Col}(\omega, \kappa)$-generic over M.

Let $N = HOD^*_{M[G]}$. By Theorem 3, N is a countable transitive $\in$-model of $\mathsf{ZF}+\mathsf{DC}$, so it remains to show that N is also a model of LM.

Let $S \in N$ with $S \subseteq \mathbb{R}^N$. We must prove that $(S \text{ is Lebesgue measurable})^N$.

By definition of HOD^* there exists a formula $\psi(x, y)$ in the language of set theory and $f \in M[G]$ with $f : \omega \to On^M$ such that $\psi(f, S)^{M[G]}$ and S is unique in $M[G]$ with this property. Now let $\phi(x, z)$ be the formula $\exists y\,(z \in y \wedge \psi(x, y))$, so that for all $r \in M[G]$, $r \in S$ iff $\phi(f, r)^{M[G]}$. Then by viewing real numbers as functions $\omega \to \{0, 1\}$ and using a two-variable version of Theorem 5, we obtain a formula $\widetilde{\phi}(x, y)$ such that for all $r \in M[G]$, $r \in S$ iff $\widetilde{\phi}(f, r)^{M[f][r]}$. Thus S is Solovay over $M[f]$.

It follows by Corollary 2 that there exists $d \in \mathrm{BC}^{M[f]}$ such that $S \triangle A_d \subseteq \bigcup\left\{A_c : c \in \mathrm{BC}^{M[f]} \text{ and } A_c \text{ is null}\right\}$. Note that $d \in M[G]$ since $M[f] \subseteq M[G]$.

Now by Theorem 4, $\left(2^{\aleph_0}\right)^{M[f]} < \kappa$. But by Corollary 1, $\kappa \leq \aleph_1^{M[G]}$. It follows that $\left(\left(2^{\aleph_0}\right)^{M[f]} \text{ is countable}\right)^{M[G]}$ and hence $\left(\mathrm{BC}^{M[f]} \text{ is countable}\right)^{M[G]}$. So we may let $e \in \mathrm{BC}^{M[G]}$ be the Borel code describing the countable union $\bigcup\left\{A_c : c \in \mathrm{BC}^{M[f]} \text{ and } A_c \text{ is null}\right\}$. Crucially, $e \in M[G]$. Then since $\mathrm{BC}^{M[f]}$ is countable, it follows by countable additivity of Lebesgue measure that A_e is null.

Thus we have found $d, e \in \mathrm{BC}^{M[G]}$ such that $S \triangle A_d \subseteq A_e$ and A_e is null. Note that we could use Fact 1 to deduce from this that S is Lebesgue measurable.

To deduce that $(S \text{ is Lebesgue measurable})^N$, recall that Borel codes are functions $\omega \to \omega$. Therefore by Lemma 2, $d, e \in \mathrm{BC}^N$. Here we have used part 1 of Theorem 6. Next, since $S \subseteq \mathbb{R}^N$, it follows by part 2 of Theorem 6 that $S \triangle A_d^N \subseteq A_e^N$. Finally, since A_e is null, it follows by part 4 of Theorem 6 that $(A_e \text{ is null})^N$. Thus by applying Fact 1 in N, $(S \text{ is Lebesgue measurable})^N$. □

3 Necessity of the inaccessible

Recall the question we were originally hoping to settle: is $\mathsf{ZF}+\mathsf{DC}+\mathsf{LM}$ consistent? Theorem 1 showed us that the answer is yes, assuming that $\mathsf{ZFC}+\mathsf{I}$ is consistent. It is natural to ask whether this hypothesis can be weakened.

This chapter is devoted to proving, in outline, the following converse to Theorem 1, which shows that it cannot be weakened.

Theorem 7. *If* $\mathsf{ZF}+\mathsf{DC}+\mathsf{LM}$ *is consistent, then* $\mathsf{ZFC}+\mathsf{I}$ *is consistent.*

To prove Theorem 7, we will prove the following in $\mathsf{ZF}+\mathsf{DC}$. Here, L is Gödel's constructible universe.

Theorem 8 (Shelah [7]). *Suppose every set of reals is Lebesgue measurable. Then (κ is a strongly inaccessible cardinal)^L, where $\kappa = \aleph_1$.*

Remark. Of course it cannot be the case that ($\aleph_1$ is a strongly inaccessible cardinal)^L. But we need not have $\aleph_1^L = \aleph_1$, while of course by κ^L we mean κ.

To prove Theorem 7 from this, we use the fact that L is a "class model" of ZFC in the sense that for each axiom ϕ of ZFC, ϕ^L is a theorem of ZF [2, pp. 175–190].

Proof of Theorem 7 from Theorem 8. By Theorem 8, I^L is a theorem of $\mathsf{ZF} + \mathsf{DC} + \mathsf{LM}$. But for each axiom ϕ of ZFC, ϕ^L is also a theorem of $\mathsf{ZF} + \mathsf{DC} + \mathsf{LM}$. It follows that for any theorem ψ of $\mathsf{ZFC} + \mathsf{I}$, ψ^L is a theorem of $\mathsf{ZF} + \mathsf{DC} + \mathsf{LM}$ by relativising the proof to L. In particular, if $\mathsf{ZFC} + \mathsf{I}$ is inconsistent then so is $\mathsf{ZF} + \mathsf{DC} + \mathsf{LM}$. □

For the rest of this chapter, then, we work in $\mathsf{ZF} + \mathsf{DC}$ unless stated otherwise.

3.1 Constructible closure

We follow the approach of Semmes [6] and begin by dealing with the part of the proof of Shelah's theorem concerning Gödel's constructible universe. In doing so we shall reduce Shelah's theorem to the following result.

Theorem 9 (Raisonnier [8]). *If there is an uncountable well-ordered set of reals, then there is a non-measurable set of reals.*

To prove Shelah's theorem from this, we require the notion of constructible closure. The following definition is from Kanamori [3, p. 34].

Notation. Denote by def the definable power set operation used to define L.

Definition. Let x be a set. The *constructible closure of* x is

$$L(x) := \bigcup_{\alpha \in On} L_\alpha(x),$$

where $L_\alpha(x)$ is defined recursively for $\alpha \in On$ by:

- $L_0(A) = \mathrm{TC}(\{x\})$;
- $L_{\alpha+1}(A) = \mathrm{def}(L_\alpha(x))$;
- $L_\delta(x) = \bigcup_{\alpha<\delta} L_\alpha(x)$ for $\delta \in On$ a non-zero limit.

Thus $L(\emptyset) = L$. It can be shown that $L(x)$ is an inner model, meaning that it is a transitive class $\in$-model of ZF containing every ordinal, and moreover it is the smallest (by inclusion) inner model containing x [3, p. 34]. Furthermore, if $L(x)$ contains a well-ordering of $\mathrm{TC}(\{x\})$, then, like L, there is a class well-ordering of $L(x)$ and hence AC also holds in $L(x)$. Note that this occurs when $x \in \mathbb{R}$, for in this case, viewing real numbers as subsets of ω, $\mathrm{TC}(\{x\})$ is either finite or equal to $\omega \cup \{x\}$.

Lastly, we also need the fact that the generalised continuum hypothesis holds in L [2, pp. 190–191].

We are now ready to prove Shelah's theorem from Raisonnier's theorem. We use ideas from Bekkali [5, p. 8].

Proof of Theorem 8 from Threorem 9. Let $\kappa = \aleph_1$, and suppose for contradiction that (κ is not strongly inaccessible)^L. Clearly (κ is a regular cardinal)^L since $\aleph_1$ is a regular cardinal, so (κ is not a strong limit)^L. But strong limit cardinals are those uncountable cardinals of the form $\beth_\delta$ with $\delta \in On$ a limit ordinal, so since the generalised continuum hypothesis holds in L, (κ is not a limit cardinal)^L. So there exists a $\alpha < \kappa$ such that (κ is the cardinal successor to α)^L. In particular, if $\alpha < \beta < \kappa$ then L contains a bijection $\alpha \to \beta$.

Now since $\alpha < \aleph_1$, there is a well-ordering of ω of order type α, which, viewed as a subset of $\omega \times \omega$, has a corresponding $x \in \mathbb{R}$ satisfying (α is countable)$^{L(x)}$. But if $\alpha < \beta < \kappa$ then $L\left(x\right) \supseteq L$ contains a bijection $\alpha \to \beta$ and hence (β is countable)$^{L(x)}$. Thus $(\kappa = \aleph_1)^{L(x)}$.

But AC holds in $L\left(x\right)$, so $L\left(x\right)$ contains a set of reals of with a well-ordering of order type $\aleph_1^{L(x)} = \aleph_1$. Thus there is an uncountable well-ordered set of reals and so by Theorem 9, there is a non-measurable set of reals. Contradiction. □

3.2 Cantor space

Rather than prove Raisonnier's theorem in $\mathbb{R}$ directly, we work in a related measure space known as "Cantor space", which can be embedded into $\mathbb{R}$ via the familiar Cantor set.

Definition. The *Cantor ternary set* $C_{1/3} \subseteq \mathbb{R}$ is the set obtained from $[0,1]$ by successively removing the open middle third from every remaining interval. That is,

$$C_{1/3} := [0,1] \setminus \bigcup_{n=1}^{\infty} \bigcup_{m=0}^{3^{n-1}-1} \left(\frac{3m+1}{3^n}, \frac{3m+2}{3^n}\right).$$

Cantor space is the topological space

$$\mathcal{C} := {}^{\omega}\left\{0,1\right\}$$

with the product topology induced by the discrete topology on $\{0,1\}$. We identify $\mathcal{C}$ and $\mathcal{P}\left(\omega\right)$ so that for $x \in \mathcal{C}$ and $n \in \omega$, $x\left(n\right) = 1$ iff $n \in x$.

The following result is as an easy exercise.

Proposition 10. *If $C_{1/3}$ is given the subspace topology, then the base-3 expansion map $\mathcal{C} \to C_{1/3}$ given by*

$$x \mapsto \sum_{n=0}^{\infty} \frac{2x\left(n\right)}{3^{n+1}}$$

is a homeomorphism. □

It might appear that we should search for a non-measurable subset of $C_{1/3}$ by studying $\mathcal{C}$ and then applying Proposition 10. But this is doomed to failure, because in fact $C_{1/3}$ is null and therefore so are all of its subsets. To see this, observe that at each stage in the construction of $C_{1/3}$ we remove $\frac{1}{3}$ of the remaining measure, and apply countable additivity.

We therefore need a different way of embedding $\mathcal{C}$ into $\mathbb{R}$. The following definition is based on Wikipedia [9].

Definition. The *fat Cantor set* $C^* \subseteq \mathbb{R}$ is

$$C^* := [0,2] \setminus \bigcup_{n=0}^{\infty} C_n,$$

where for each $n \in \omega$, C_n consists of an open interval of width $\frac{1}{2^{2n+1}}$ taken from the middle of each interval of $[0,2] \setminus \bigcup_{m=0}^{n-1} C_m$.

In a similar fashion to Proposition 10, there is a natural homeomorphism $f : C^* \to \mathcal{C}$. Defining the Borel algebra $\mathcal{B}^*$ of $\mathcal{C}$ analogously to the Borel algebra $\mathcal{B}$ of $\mathbb{R}$, it follows easily that

$$\mathcal{B}^* = \left\{ f\left(B \cap C^*\right) : B \in \mathcal{B} \right\}.$$

Moreover, for each $n \in \omega$ the measure of C_n is $2^n . \frac{1}{2^{2n+1}} = \frac{1}{2^{n+1}}$ and hence the total measure of C^* is $2 - \sum_{n=0}^{\infty} \frac{1}{2^{n+1}} = 1$. So the measure induced by Lebesgue measure on $\mathcal{C}$ via f is non-trivial.

This induced measure is also called Lebesgue measure and denoted by μ. One can show that μ is equal to the so-called completion of the product measure on $\mathcal{C}$ induced by the uniform probability measure on $\{0,1\}$. This is the probability measure that is usually used to describe an infinite sequence of coin tosses. Thus we have reduced the problem of finding a non-measurable set of reals to the problem of finding a non-measurable subset of $\mathcal{C}$ under this measure.

We now introduce the measure-theoretic properties of $\mathcal{C}$ required for the proof of Raisonnier's theorem.

Definition. We say $A \subseteq \mathcal{C}$ is a *tail set* iff for every $x \in A$, if $y \in \mathcal{C}$ with $x(n) = y(n)$ for all but finitely many $n \in \omega$, then $y \in A$.

Let $x \in \mathcal{C}$ and $n \in \omega$. The *ball of codimension n about x* is

$$B_n(x) = \{ y \in \mathcal{C} : x(m) = y(m) \text{ for all } m \in \{0, 1, \ldots, n-1\} \}.$$

Let $A \subseteq \mathcal{C}$ be measurable and $x \in \mathcal{C}$. The *density of A at x* is

$$d_A(x) = \lim_{n \to \infty} \frac{\mu(A \cap B_n(x))}{\mu(B_n(x))},$$

if the limit exists.

The following list of results is based on Semmes [6, p. 5]. The proofs can be found in any good text on measure theory.

Fact 3. *Let $A \subseteq \mathcal{C}$ be measurable.*

1. *(*Kolmogorov's zero–one law*) If A is a tail set, then $\mu(A) = 0$ or 1.*
2. *(*Lebesgue's density theorem*) $A \triangle \{x \in \mathcal{C} : d_A(x) = 1\}$ is null.*
3. *(*Fubini's theorem*) Let $f : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ be a homeomorphism induced by a bijection $\omega + \omega \to \omega$. Then A is null iff $\{x \in \mathcal{C} : \mu(\{y \in \mathcal{C} : f(x,y) \in A\}) > 0\}$ is null.*

We conclude this section with an instructive example of a non-measurable subset of $\mathcal{C}$ that can be constructed using AC. The following is based on Bekkali [5].

Example. Assume AC. Recall that by Zorn's lemma, any filter on ω[14] can be extended to a maximal filter, also known as an ultrafilter, which contains exactly one member of the pair $\{x, \omega \setminus x\}$ for each $x \subseteq \omega$. Extend the cofinite filter $\{x \subseteq \omega : \omega \setminus x \text{ is finite}\}$ to an ultrafilter U. The result is known as a non-principal ultrafilter on ω and does not contain any finite sets.

Suppose for contradiction that U, viewed as a subset of $\mathcal{C}$, is measurable. First we claim that U is a tail set. To see this, suppose $x \in U$ and $y \in \mathcal{C}$ differs from x in only finitely many places. Then $\omega \setminus y \notin U$, else $x \cap (\omega \setminus y)$ is a finite set lying in U. So $y \in U$ by maximality of U. This proves the claim. It follows by Kolmogorov's zero–one law that $\mu(U) = 0$ or 1.

But the map $g : \mathcal{P}(\omega) \to \mathcal{P}(\omega)$ defined by $g(x) = \omega \setminus x$ induces a measure-preserving transformation on $\mathcal{C}$, since it corresponds to a reflection of C^* about the point 1. Moreover, $\mathcal{P}(\omega)$ is equal to the disjoint union $U \cup g(U)$ by maximality of U. Hence we must have $\mu(U) = \mu(g(U)) = \frac{1}{2}$. Contradiction.

3.3 Rapid filters and outline of proof of Raisonnier's theorem

Motivated by the above example, we attempt to approximate the notion of a non-principal ultrafilter by defining a "rapid filter". This is closely related to the idea of a dominating family. The following definition is based on Semmes [6, p. 8].

Definition. To each $x \subseteq \omega$ associate the sequence $(x_n)_{n \in \omega}$ listing the members of x in ascending order.

A filter F on ω is *rapid* iff for every $y \subseteq \omega$ there exists $x \in F$ such that for all $n \in \omega$, $x_n \geq y_n$.

Observe that if F is a rapid filter on ω then F does not contain any finite sets, else the set of minimal elements of members of the filter is bounded. Observe also that F is uncountable by a diagonal argument. Thus rapid filters approximate non-principal ultrafilters in the sense that a rapid filter is a fairly "large" subset of a non-principal ultrafilter. Moreover, we will be able to adapt the argument used for non-principal ultrafilters to show that rapid filters are non-measurable.

To construct a rapid filter from an uncountable well-ordered set of reals we require a couple more pieces of terminology. The following definition is based on Semmes [6, p. 10].

Definition. Let S be a collection of finite subsets of ω.

We say that S *captures* $X \subseteq \mathcal{C}$ iff for every $x \in X$, $x \cap \{0, 1, \ldots, n-1\} \in S$ for all sufficiently large $n \in \omega$.

We say that S *splits on* $n \in \omega$ iff there exists $s \subseteq \{0, 1, \ldots, n-1\}$ such that $s, s \cup \{n\} \in S$.

Our rapid filter will come from taking X to be a subset of $\mathcal{C}$ of cardinality $\aleph_1$ in the following definition, which is also based on Semmes [6, p. 10].

[14] A filter on a set X means a filter of the poset $\mathcal{P}(X) \setminus \{\emptyset\}$ ordered by inclusion.

Definition. Let $X \subseteq \mathcal{C}$ be uncountable. Define

$$F(X) := \{x \in \mathcal{C} : \text{ there is a collection } S \text{ of finite subsets of } \omega \text{ such that } S \text{ captures } X \text{ and splits only on members of } x\}.$$

It is an exercise to show that $F(X)$ is a filter for each uncountable $X \subseteq \mathcal{C}$. The uncountability of X is used to show that $\emptyset \notin F(X)$ [6, p. 10].

We are now ready to outline the proof of Raisonnier's theorem. For the full proof, we refer the reader to Semmes [6] or Bekkali [5, pp. 1–8]. Here, we follow the approach of Semmes.

Outline of proof of Theorem 9. Suppose there is an uncountable well-ordered set of reals. Then in particular there is a set of reals of cardinality $\aleph_1$. Using a bijection $\mathbb{R} \to \mathcal{P}(\omega)$ we obtain $X \subseteq \mathcal{C}$ of cardinality $\aleph_1$. Now prove the following three results.

1. *If every subset of $\mathcal{C}$ is measurable, then the union of $\aleph_1$ null subsets of $\mathcal{C}$ is null.* First show that we may assume without loss of generality that the union is a tail set. Then use Fubini's theorem to show that it cannot have measure 1. It follows by Kolmogorov's zero–one law that it is null.

2. *Rapid filters are non-measurable.* Let F be a rapid filter and suppose for contradiction that F is measurable. Use Lebesgue's density theorem to show that F meets every closed subset of $\mathcal{C}$ of positive measure, and deduce that $\mu(F) = 1$. Now recall from the example of the non-principal ultrafilter that the map $g : \mathcal{P}(\omega) \to \mathcal{P}(\omega)$ defined by $g(x) = \omega \setminus x$ is measure-preserving. So since F is a filter, F and $g(F)$ are disjoint sets of equal measure and hence $\mu(F) \leq \frac{1}{2}$. Contradiction.

3. *If the union of $\aleph_1$ null subsets of $\mathcal{C}$ is null, then $F(X)$ is rapid.* First use the hypothesis to show that for every $y \subseteq \omega$ there exists a collection S of finite subsets of ω such that S captures X and $|\{s \in S : s \subseteq \{0, 1, \ldots, y_n - 1\}\}| \leq n$ for all $n \in \omega$. This is the heart of the proof. Deduce from this that $F(X)$ is rapid.

Together these results show that if every subset of $\mathcal{C}$ is measurable, then $F(X)$ is non-measurable. Therefore there is a non-measurable subset of $\mathcal{C}$. By embedding this into the fat Cantor set C^* we obtain a non-measurable set of reals. □

4 Conclusion

We began with the question: is it consistent with $\mathsf{ZF}+\mathsf{DC}$ that every set of reals is Lebesgue measurable? Through Theorems 1 and 7 we have seen that $\mathsf{ZF} + \mathsf{DC} + \mathsf{LM}$ is consistent iff $\mathsf{ZFC} + \mathsf{I}$ is consistent. The question remains: does the consistency of $\mathsf{ZFC} + \mathsf{I}$ follow from the consistency of ZFC?

To answer this question, recall that if κ is a strongly inaccessible cardinal, then V_κ is a model of ZFC [1, p. 110]. It follows that $\mathsf{ZFC} + \mathsf{I}$ proves $\mathrm{Con}(\mathsf{ZFC})$. So if $\mathsf{ZFC} + \mathsf{I}$ is consistent, then ZFC cannot prove $\mathrm{Con}(\mathsf{ZFC}) \Rightarrow \mathrm{Con}(\mathsf{ZFC} + \mathsf{I})$, else $\mathsf{ZFC} + \mathsf{I}$ would prove $\mathrm{Con}(\mathsf{ZFC} + \mathsf{I})$, contrary to Gödel's second incompleteness theorem.

Nonetheless, the consistency of $\mathsf{ZFC} + \mathsf{I}$ is a very "believable" statement for a couple of reasons. Firstly we have an analogy with the axiom of infinity [2, p. 58] [3, p. XVI]: $\aleph_0$ can be viewed as the limit of what we can obtain from finite cardinals in $\mathsf{ZFC} - \mathsf{Infinity}$, with $V_{\aleph_0}$ being a model of $\mathsf{ZFC} - \mathsf{Infinity}$; likewise, a strongly inaccessible cardinal κ can be viewed as the limit of what we can obtain from smaller cardinals in ZFC, with V_κ being a model of ZFC. Secondly, this analogy extends much further in the theory of large cardinals, with many large cardinal axioms implying the consistency of $\mathsf{ZFC} + \mathsf{I}$, and no evidence of an inconsistency even for far stronger axioms.

Thus we propose the following answer to the original question.

Answer. Assuming ZFC is consistent, we cannot prove from this that $\mathsf{ZF} + \mathsf{DC} + \mathsf{LM}$ is consistent by methods formalisable in ZFC.

However, the consistency of $\mathsf{ZF} + \mathsf{DC} + \mathsf{LM}$ does follow from the "believable" statement that $\mathsf{ZFC} + \mathsf{I}$ is consistent.

4.1 Further results

We conclude by mentioning various related results.

Notation. Denote by B the statement that every subset of reals has the Baire property, i.e. differs from an open set by a countable union of nowhere dense sets.

Denote by P the statement that every uncountable set of reals has a perfect subset, i.e. a non-empty closed subset with no isolated points.

Denote by W the statement that there is a weakly inaccessible cardinal.

Just as for LM, neither B nor P hold in ZFC, but both hold in Solovay's model. Hence if $\mathsf{ZFC} + \mathsf{I}$ is consistent then so is $\mathsf{ZF} + \mathsf{DC} + \mathsf{LM} + \mathsf{B} + \mathsf{P}$ [3, pp. 132–141]. Much like LM, if $\mathsf{ZF} + \mathsf{P}$ is consistent then so is $\mathsf{ZFC} + \mathsf{I}$ [13]. In contrast, however, the consistency of $\mathsf{ZF} + \mathsf{DC} + \mathsf{B}$ does follow just from the consistency of ZFC [7].

Next we observe that even though a strongly inaccessible cardinal need not be weakly inaccessible, $\mathsf{ZFC} + \mathsf{W}$ is consistent iff $\mathsf{ZFC} + \mathsf{I}$ is consistent. To see the "if" statement, simply observe that in ZFC, any strongly inaccessible cardinal is weakly inaccessible. The converse follows from the fact that in ZFC, if κ is a weakly inaccessible cardinal, then (κ is a strongly inaccessible cardinal)^L since the generalised continuum hypothesis holds in L.

We return now to the issue of Lebesgue measurability. Solovay also showed that if ZFC is consistent then $\mathsf{ZF} + \mathsf{DC} +$ "there is a monotone, countably additive and translation invariant extension of Lebesgue measure to all subsets of the reals" is consistent [14]. Since this result does not mention the existence of a strongly inaccessible cardinal, this is perhaps a more satisfying solution to the measure problem described in the introduction.

Perhaps an even more satisfying solution was given by Shelah and Woodin [15], who showed that if there is a supercompact cardinal (see Kanamori [3, p. 298]), then the constructible closure of the reals $L(\mathbb{R})$ is a class model of $\mathsf{ZF} + \mathsf{DC} + \mathsf{LM} + \mathsf{B}$. In some sense, if Solovay's theorem says that every "definable" set of reals is Lebesgue measurable, then this result says that every "transfinitely definable" (or as Shelah and Woodin put it, "reasonably definable") set of reals is Lebesgue measurable (and has the Baire property). However, the large cardinal hypothesis here is strictly stronger than I.

Finally, for those that are content that ZF+DC+LM is consistent, we propose the related question: how strong a form of the axiom of choice is needed to construct a non-measurable set in ZF? Theorem 9 demonstrates that DC+"there is an uncountable well-ordered set of reals" is enough. From this it follows that DC+"there is no partition of $2^{\aleph_0}$ into more than $2^{\aleph_0}$ pairwise disjoint non-empty sets" is enough (see Appendix B). The example at the end of section 3.2 shows that the existence of a non-principal ultrafilter on ω is enough. We refer the reader to Herrlich [11, p. 156] for a wide variety of other statements with this property. These include the existence of a basis for $\mathbb{R}$ as a vector space over $\mathbb{Q}$, the axiom of choice for pairs, and the Hahn-Banach theorem.

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A Sketch of proof of the factor lemma

As in section 2.2, we work in ZFC and assume that M is a countable transitive $\in$-model of ZFC and that $\kappa \in M$ with $(\kappa \text{ is a strongly inaccessible cardinal})^M$. Let G be a $\mathrm{Col}(\omega, \kappa)$-generic filter over M.

Recall the following lemma.

Lemma 5 (factor lemma). *Let $f \in M[G]$ with $f : \omega \to On^M$. Then there is a* $\mathrm{Col}(\omega, \kappa)$-*generic filter H over $M[f]$ such that $M[G] = M[f][H]$.*

The full proof can be found in Kanamori [3, pp. 129–131] or Jech [2, pp. 516–518]. Here, we sketch Kanamori's proof, which begins with the following lemma.

Lemma 7. *Let P be a separative forcing poset with maximal element 1, and let $\alpha \in On$ with $|\alpha| \geq |P|$. Suppose*

$$1 \Vdash (\exists f) \left(f : \omega \to \alpha \text{ is surjective and } f \notin \check{V} \right).$$

Then there is an order-isomorphism between a dense subset of $\mathrm{Col}(\omega, \{\alpha\})$ *and a dense subset of P. Hence* $\mathrm{Col}(\omega, \{\alpha\})$ *and P have the same generic extensions.*

Sketch of proof. The dense subset of $\mathrm{Col}(\omega, \{\alpha\})$ that we shall use is

$$D = \{p \in \mathrm{Col}(\omega, \{\alpha\}) : \mathrm{dom}(p) = \{\alpha\} \times n \text{ for some } n \in \omega\}.$$

The poset embedding $g : D \to P$ is defined recursively, as follows. First define $g(\emptyset) := 1$. Then, having defined $g(p)$ for some $p \in D$ with $\mathrm{dom}(p) = \{\alpha\} \times n$, where $n \in \omega$, choose a maximal antichain $\{a_\beta : \beta < \alpha\}$ of P below $g(p)$ such that for all $\beta < \alpha$, there exists $q \in P$ such that $a_\beta \Vdash \dot{g}(n) = \check{q}$. Here, $\dot{g}$ is a P-name such that $1 \Vdash (\dot{g} : \omega \to \dot{G} \text{ is surjective})$, where $\dot{G}$ is the canonical P-name for a generic object. Then define $g(p \cup \{((\alpha, n), \beta)\}) := a_\beta$. ☐

To complete the sketch of Kanamori's proof, we draw on Lemma 4, which relates the intermediate extension to a forcing extension. We will also need to re-use the argument from the last paragraph of the proof of Lemma 4.

Sketch of proof of Lemma 5. By Lemma 4, there exists $\alpha < \kappa$ such that $M[f] \subseteq M[G \cap \mathrm{Col}(\omega, \{\alpha\})]$, where $G \cap \mathrm{Col}(\omega, \{\alpha\})$ is a $\mathrm{Col}(\omega, \alpha)$-generic filter over M.

Let $G_0 = G \cap \mathrm{Col}(\omega, \alpha)^M$, $G_1 = G \cap \mathrm{Col}(\omega, \{\alpha\})^M$ and $G_2 = G \cap \mathrm{Col}(\omega, \kappa \setminus (\alpha + 1))^M$.

First, one can find a separative partial order $P \in M[f]$ and a P-generic filter H_0 over $M[f]$ such that $M[f][H_0] = M[G_0]$. Let $Q = P \times \mathrm{Col}(\omega, \{\alpha\})^M$. One can then apply Lemma 7 to Q to obtain a $\mathrm{Col}(\omega, \{\alpha\})$-generic filter H_1 over $M[f]$ such that $M[f][H_1] = M[f][H_0][G_1]$.

Next, one can apply Lemma 7 to $\mathrm{Col}(\omega, \alpha + 1)^M$ to obtain a $\mathrm{Col}(\omega, \alpha + 1)$-generic filter H_2 over $M[f]$ such that $M[f][H_2] = M[f][H_1]$. By repeatedly applying the argument from the last paragraph of the proof of Lemma 4, we find that $M[G] = M[G_0][G_1][G_2] = M[f][H_0][G_1][G_2] = M[f][H_1][G_2] = M[f][H_2][G_2] = M[f][H]$, where $H = H_2 \cup G_2$ is a $\mathrm{Col}(\omega, \kappa)$-generic filter over $M[f]$. ☐

B Fun corollary to Theorem 9

Recall the following theorem of $\mathsf{ZF} + \mathsf{DC}$ (Theorem 9): if there is an uncountable well-ordered set of reals, then there is a non-measurable set of reals. From this we obtain the following theorem of $\mathsf{ZF} + \mathsf{DC}$, which illustrates a counterintuitive consequence of LM.

Corollary 3. *Suppose every set of reals is Lebesgue measurable. Then there is a partition of* $2^{\aleph_0}$ *into more than* $2^{\aleph_0}$ *pairwise disjoint non-empty sets.*

The proof is due to a pseudonymous MathOverflow user [12].

Proof. First note that $|\mathcal{P}(\omega \times \omega)| = 2^{\aleph_0}$. Then define the equivalence relation $\sim$ on $P(\omega \times \omega)$ by $X \sim Y$ iff either $X = Y$ or X and Y are isomorphic well-orderings of ω. Since for example any $X \subseteq \omega \times \omega$ containing both $(0, 1)$ and $(1, 0)$ is not a well-ordering of ω, there are $\aleph_1 + 2^{\aleph_0}$ equivalence classes.

To see that $\aleph_1 + 2^{\aleph_0} > 2^{\aleph_0}$, observe that there is clearly an injection $2^{\aleph_0} \to \aleph_1 + 2^{\aleph_0}$, but since every set of reals is Lebesgue measurable, by Theorem 9 there is no injection $\aleph_1 \to 2^{\aleph_0}$ and hence no injection $\aleph_1 + 2^{\aleph_0} \to 2^{\aleph_0}$. $\square$